MATEMATIČKI FAKULTET – SKOPJE

ALGEBRAIC CONFERENCE

Skopje 1980

1980

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MATEMATIČKI FAKULTET - SKOPJE

A L G E B R A I C   C O N F E R E N C E

S k o p j e   1980

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P R E F A C E

The Conference of algebraists from Belgrade, Novi Sad and Skopje took place in Skopje on February 1-3, 1980. There were 27 official participants with 20 reports from:

I. Faculty of Sciences, Belgrade: 5 participants with 5 reports (the papers Nos 1-6 in the contents);

II. Mathematical Institute, Belgrade: 5 participants with 4 reports (Nos 7-10);

III. Faculty of Sciences, Novi Sad: 6 participants with 4 reports (Nos 11-15);

IV. Faculty of Mathematics, Skopje: 10 participants with 7 reports (Nos 16-21).

Beside the official participants, some other mathematicians attended the Conference.

The representatives gave short reviews of the teaching algebra in their institutions and in the discussion some suggestions were given for improvement of the algebra teaching from the methodological and subject-matter point of views.

In a special debate it was unanimously welcomed the initiative for organizing this Conference. Establishing that such meetings are very useful and significant for interchanging scientific informations, direct contacts and fruitful collaboration on many fields, the participants decided:

-to include algebraists from all over Yugoslavia;

-the next Conference to be organized by the algebraists of the Faculty of Sciences from Novi Sad in 1981;

-the work on the Conference to be done in sections, following a particular programme;

-the reports of the Conference to be published in a special book.

This book of proceedings is a result of the above decisions. The papers are published here in the order in which they were presented during the Conference. The papers No 6 and No 15 (in the contents), due to the absence of the authors, were not communicated, and the paper No 21 is a joint of two reports presented on the Conference by the two authors separately.

Skopje

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CONTENTS

Algebraic conference
S k o p j e 1980

REMARKS ON SOME CONSTRUCTIONS IN ALGEBRA

Slaviša B. Prešić

1. We start with an example. It is well known [1], p. 185 that every universal algebra can be embedded into certain semigroup. For instance, according to this theorem, in case of the groupoid determined by the table

(1)

o	a	b
a	a	b
b	a	a

we have the following assertion:

(2) There exists a set $S \supset \{a,b\}$, an operation $\ast : S^2 \to S$ and $c \in S$ such that $(S,\ast)$ is a semigroup and the equality

$$x o y = (c \ast x) \ast y$$

is true for all $x,y \in \{a,b\}$.

We describe a construction of such a semigroup $(S,\ast)$, whose elements will be generated by a,b and one new element c. In fact this construction is an ilustration of some general ideas we are going to describe after that.

At first extend the set $\{a,b\}$ by a new element, c say. Further, by Term(a,b,c,$*$), where $*$ is a binary operation symbol, we denote the set of all terms built up from a,b,c and $*$ (without any variables).

In connection with (2) form the following set H(=(3) $\cup$ (4) $\cup$ (5)):

(3) $(c*a)*a = a, \quad (c*a)*b = b, \quad (c*b)*a = a, \quad (c*b)*b = a,$

i.e. all equalities of the form

$$x \circ y = (c*x)*y \qquad (x,y \in \{a,b\})$$

(4) $$a \neq b$$

(5) $(t_1*t_2)*t_3 = t_1*(t_2*t_3)$ for all $t_i \in \mathrm{Term}(a,b,c,*)$.

Further let Ax(=) be the set of the following formulas (the equality axioms):

(6) $t_1=t_1, \quad t_1=t_2 \Rightarrow t_2=t_1, \quad t_1=t_2 \wedge t_2=t_3 \Rightarrow t_1=t_3$

$$t_1 = t_2 \wedge t_3 = t_4 \Rightarrow t_1*t_3 = t_2*t_4 \quad (t_i \in \mathrm{Term}(a,b,c,*)).$$

Obviously any normal model[1] of the set $H \cup Ax(=)$ (of the set H as well) determines the required semigroup $(S, \circledast)$.

We point out that the set $H \cup Ax(=)$ is a set of basic[2] Horn formulas.

Besides this, in algebra, there are many other problems which can be expressed by means of certain sets of basic Horn formulas. For instance, the problems of isomorphic embeddings, constructions of free algebras and so on.

We describe one method which is often useful for solving such problems. This method is partly original. The paper is deeply connected with [2].

At first we construct the required semigroup $(S, \circledast)$.

In the first step we search for some quasi-algebra [2], which follows from the set $H \cup Ax(=)$. To achieve this, define the equivalence relation $\sim$ (of Term(a,b,$*$)) as follows:

(7) $t_1 \sim t_2$ iff $H \cup Ax(=) \vdash t_1 = t_2$.

Using this definition it is easy to prove that:

Each member t of Term(a,b,c,$*$) is equivalent to at least one term t' having the property:

If t' contains[3] the symbol c, then after every occurrence of c there is at most one occurrence of a or b.

Let M be the set of all t', where $t \in \mathrm{Term}(a,b,c,*)$. The members t' of M are called <u>markers</u>. For instance

$$a,\ aa,\ c,\ aca,\ ccac \in M, \quad \text{but} \quad cab \notin M.$$

Let m_1, m_2 be any two markers and let $\overline{m_1m_2}$ be the marker which is obtained from the term m_1m_2 by using the substitutions (see (3)):

$$caa \to a, \quad cab \to b, \quad cba \to a, \quad cbb \to a.$$

It is easy to prove that $\overline{m_1m_2}$ is unique determined by m_1, m_2. For example, if m_1 = caccb, m_2 = abca, then

$$\overline{m_1m_2} = \overline{caccbabca} = \overline{cacabca}$$
$$cba \to a$$
$$= \overline{cabca} = bca$$
$$cab \to b.$$

Denote by Q the set of all equalities of the form:

$$m_1m_2 = \overline{m_1m_2}.$$

This set is a quasi-algebra.

In the second step we check whether the quasi-algebra Q is incontractible. Using the way described in [2] it is easy to see that Q is incontractible, i.e. that

$$\text{not} \quad Q \cup Ax(=) \vdash m_1 = m_2$$

where m_1, m_2 are some two different members of the set M. The set Q is a consequence of the set $H \cup Ax(=)$. That is why the equivalence

(8) $\quad H \cup Ax(=) \quad$ <u>equiv</u>. $\quad H \cup Ax(=) \cup Q$

is true.

In the third step we replace each term t of each formula of the set H by the corresponding marker t´. In such a way from the set H we obtain a new set H_Q - so called <u>the reduct of the set H modulo the quasi-algebra Q</u>.

Obviously the equivalence

(9) $\quad H \cup Ax(=) \cup Q \quad$ <u>equiv</u>. $\quad H_Q \cup Ax(=) \cup Q$

is true. From (8), (9) if follows that

(10) $\quad H \cup Ax(=) \quad$ <u>equiv</u>. $\quad H_Q \cup Ax(=) \cup Q$.

We now "calculate" H_Q. For the set (3) we have

$(3)_Q = \{a=a,\ b=b,\ a=a,\ a=a\}$.

It is not quite easy to find $(5)_Q$, i.e. the set

$$(5)_Q = \{\overline{\overline{m_1 m_2} m_3} = \overline{m_1 \overline{m_2 m_3}} \mid m_1, m_2, m_3 \in M\}.$$

The result is

$$(5)_Q = \{m = m \mid m \in M\}.$$

In other words the quasi-algebra Q satisfies the associative law. For the set (4) we have

$$(4)_Q = \{a \neq b\},$$

because a,b are markers and therefore: a´ is a, b´ is b.

As $H_Q = (3)_Q \cup (4)_Q \cup (5)_Q$ we have the following result (for H_Q):

$$H_Q = \{a \neq b\} \cup \{m = m \mid m \in M\}.$$

For the set H_Q we note the following:

(11) The reduct H_Q contains neither any equality $m_1 = m_2$ between two different[4)] markers nor any member of the form $m \neq m$.

From this and (10) it follows the following equivalence

(12) $\quad H \cup Ax(=) \quad$ <u>equiv</u>. $\quad Ax(=) \cup Q \cup \{a \neq b\}$.

The set $Q \cup \{a \neq b\}$ is called <u>extended quasi-algebra</u>. Generally, if Q is any quasi-algebra and D is any set of some formulas of the form

$$m_1 \neq m_2 \qquad (m_1, m_2 \text{ are different markers}),$$

then the set $Q \cup D$ is called <u>extended quasi-algebra</u>.

From the equivalence (12) we conclude that the required semigroup is determined by any model of the extended quasi-algebra $Q \cup \{a \neq b\}$.

Obviously this extended quasi-algebra is consistent (since Q is incontractible and a,b are two different markers as well). Therefore the set $Q \cup \{a \neq b\}$ has at least one model. One of models $(S, \ast)$ (in fact <u>the marker model</u>) is determined as follows:

1° The set S is equal to the set M of all markers.

2° The operation $*$ is defined by

$$m_1 * m_2 = \overline{m_1 m_2}$$

In such a way we have completed the construction of the required semigroup $(S, *)$.

Of course to prove the assertion (2) we do not need effectively to construct one semigroup $(S, *)$ satisfying (2). Namely, to prove (2) it is sufficient to prove the existence of any model of the set $H \cup Ax(=)$, i.e. the consistency of that set.

It is easy to see that the consistency condition is equivalent to the following condition

(13) Not $H \cup Ax(=) \vdash a = b$

i.e. that the terms a,b (all members of the given groupoid (1)) are not equivalent.

Suppose the opposite, i.e.

$$H \cup Ax(=) \vdash a = b$$

which is equivalent to[5)]

$$H \vdash_{\overline{Ax(=)}} a = b,$$

i.e. the equality a=b can be derived from the set H using the equational logic. Let

(14) $a = t_1, \quad t_1 = t_2, \ldots, t_k = b$

be one of such derivations. By the substitutions

$$(c*a)*a \rightarrow (aoa), \quad (c*a)*b \rightarrow (aob),$$

$$(c*b)*a \rightarrow (boa), \quad (c*b)*b \rightarrow (bob)$$

from the derivation (14) we obtain a derivation of the equality a=b in the given groupoid, which is impossible. Consequently the condition (13) is proved.

__2.__ We are now going to study the general case. Let Ω be a given set of operation symbols and Γ a given set of constants. By Term (Ω, Γ) denote the set of all terms built up from Ω and Γ (but without variables). Further let H be a set of some (basic Horn) formulas, i.e. formulas of the form

$$\phi_1, \neg\phi_1, \phi_1 \wedge \ldots \wedge \phi_k \Rightarrow \phi_{k+1}, \quad \phi_1 \wedge \ldots \wedge \phi_k \Rightarrow \neg\phi_{k+1}$$

$(k = 1,2\ldots)$, where ϕ_i are of the form

$$t_1 = t_2 \qquad (t_1, t_2 \quad \text{Term}(\Omega, \Gamma)).$$

We also suppose that certain problem $\mathcal{P}$, like the considered, is expressed by the set H. We distinguish two cases:

To solve the problem $\mathcal{P}$ means

1° to prove that H has a model,

or 2° to construct a model of H.

The considered problem $\mathcal{P}$ belongs to the case 1°. As a matter of fact this problem is expressed by the set H $(=(3) \cup (4) \cup (5))$ but with quantifiers $(\exists S)(\exists * : S^2 \rightarrow S)$ as prefixes, i.e. $\mathcal{P}$ is exactly expressed by the "formula"

$$(\exists S)(\exists * : S^2 \rightarrow S)H.$$

Generaly in case 1° the problem $\mathcal{P}$ is expressed by H, preceding by some existencial quantifiers.

In this case to solve the problem $\mathcal{P}$ it is sufficient to prove the consistence of the set H. In the literature there are many particular ideas about it. For instance, we can often use the ideas which are similar to those used in the considered problem.

In this paper we are more interested in case 2°.

We sketch, step by step, one solving method similar to the method used in the considered example.

In the first step we search for some quasi-algebra which follows from the set $H \cup Ax(=)$. To achieve this we use the definition of the type (7).

In the second step we check whether the quasi-algebra Q is incontractible using the way described in [2].

If Q is contractible, then there are some equalities of the form

$$m_1 = m_2 \qquad (m_1,\ m_2 \text{ are different markers})$$

which follows from $Q \cup Ax(=)$. Then using such equalities we reduce the set M of all markers to some its proper subset M_1, and form a corresponding quasi-algebra, Q_1 say.

We proceede in such a way, until we obtain some quasi-algebra which is incontractible.

In the third step we form the set H_Q, the reduct of H modulo Q, by replacing each term t of every formula of the set H by the corresponding equivalent marker t'.

We also replace the formulas of the form

$$m{=}m \wedge \ldots \wedge n{=}n \Rightarrow m_1{=}m_2, \quad m = m \wedge \ldots \wedge n{=}n \Rightarrow m_1 \neq m_2$$

$$m_1 = m_2 \Rightarrow m \neq n \qquad (m_1, m_2, m, n \text{ are markers})$$

by $\qquad m_1 = m_2, \quad m_1 \neq m_2, \quad m_1 \neq m_2$

respectively

Then, similarly as in the example (see (10)) the equivalence

(15) $\qquad H \cup Ax(=) \ \underline{\text{equiv.}}\ \ H_Q \cup Ax(=) \cup Q$

is true. The members of the set H_Q are of the form

$$\phi_1,\ \neg\phi_1,\ \phi_1 \wedge \ldots \wedge \phi_k \Rightarrow \phi_{k+1},\ \phi_1 \wedge \ldots \wedge \phi_k \Rightarrow \neg\phi_{k+1}$$

$(k = 1, \ldots)$ where ϕ_i are some equalities of the form $m_1 = m_2$ (m_1, m_2 are markers).

For the set H_Q there are two possibilities

The sentence (11) is either true or false.

If (11) is false and H_Q has some member of the form $m \neq m$, then the set H is inconsistent and consequently there is no universal algebra which is a solution of the problem $\mathcal{P}$.

If (11) is false and certain equality $m_1 = m_2$ between two different markers is a member of the reduct H_Q, then using all such equalities[6)] we reduce the quasi-algebra Q and go back to the second step.

If the sentence (11) is true we form the corresponding extended quasi-algebra $Q \cup D$ (D is the set of all members of H_Q which are of the form $m_1 \neq m_2$, where m_1, m_2 are different markers). Then from (15) we conclude the following equivalence

(16) $\qquad H \cup Ax(=) \ \underline{\text{equiv.}}\ \ Q \cup D \cup Ax(=) \cup J$,

where J is a set of some implications of the form

(17) $\qquad m_1 = n_1 \wedge \ldots \wedge m_k = n_k \Rightarrow m_{k+1} = n_{k+1}$

$\qquad\qquad m_1 = n_1 \wedge \ldots \wedge m_k = n_k \Rightarrow m_{k+1} \neq n_{k+1} \quad (k=1,2,\ldots)$

and m_i, n_i $(1 \le i \le k)$ are pairwise different markers.

The marker algebra - whose members are markers and operations are defined using directly the equalities belonging to the quasi-algebra Q - is a model for Q, for $Q \cup D$, for $Q \cup D \cup J$ and also a model for H.

In this way the mentioned method is completely described. We also add the following remarks:

(i) By the described method the consistency problem for the set H is also solved.

(ii) The obtained model, i.e. the marker algebra is, in fact, a model of H generated[7] by the constants belonging to $\Gamma \cup \Omega$.

3. At the end we give one problem in which a non basic Horn set of formulas appears but which can be solved in a similar way (searching for an incontractible quasi-algebra Q which is a consequence of the correspon-ding set H, after that forming the reduct H_Q, and so on).

Let J_2 be a field with two elements 0, e defined by the tables

(18)

+	0	e
0	0	e
e	e	0

•	0	e
0	0	0
e	0	e

and $x^2 + x + e = 0$ be an equation in x.

The problem is to construct the root field for this equation. We sketch one way of solving.

Let H be the set of the following formulas ((19) $\cup \ldots \cup$ (23))

(19) $0+0=0,\quad 0+e=e,\quad e+0=e\quad e+e=0$
$0\cdot 0=0,\quad 0\cdot e=0,\quad e\cdot 0=0,\quad e\cdot e=e$
$-0=0,\quad -e=e,\quad e^{-1}=e$

(20) $$0 \neq e$$

(21) $(x+y)+z=x+(y+z),\quad x+0=x,\quad x+(-x)=0,\quad x+y=y+x$
$(x\cdot y)\cdot z=x\cdot(y\cdot z),\quad x\cdot e=x,\quad x\neq 0 \Rightarrow x\cdot x^{-1}=e,\quad x\cdot y=y\cdot x$
$x\cdot(y+z)=x\cdot y+x\cdot z$

(22) $$a^2 + a + e = 0$$

(23) $$0^{-1} = 0$$

where a is a new constant symbol and x,y,z are ele-ments of the set

(24) $$\mathrm{Term}(0,e,a,+,\cdot,-,^{-1})$$

The formulas (19) $\cup$ (20) are members of the diagram of the given field J_2, the formulas (21) are the field axioms, the formula (22) expresses that a is a solution of the given equation and the formula (23) is taken to simplify our consideration[8].

Obviously any (normal) model of these formulas determines the required[9] field.

In this case about the relation $\sim$, introduced by (as (7))

$$t_1 \sim t_2 \quad \text{iff} \quad H \cup Ax(=) \vdash t_1 = t_2,$$

we have the following assertion

Each term (a member of the set (22)) is equivalent to one of the following terms (i.e. markers)

$$0,\ e,\ a\ \ a+e.$$

For instance

$0^{-1} \sim 0$, since $0^{-1}=0 \in H$

$a^{-1} \sim a+e$, since $a(a+e) \sim e$ and $H \cup Ax(=) \vdash a \neq 0$,

which can be proved in the following way

(j) $H \cup Ax(=) \cup \{a=0\} \vdash e=0$

(jj) $H \cup Ax(=) \vdash a=0 \Rightarrow e=0$, from (j)

by the Deduction theorem

(jjj) $H \cup Ax(=) \vdash e \neq 0 \implies a \neq 0$

(jw) $H \cup Ax(=) \vdash a \neq 0$, since $e \neq 0 \in H$.

One quasi-algebra Ω, which is a consequence for the set $H \cup Ax(=)$, is determined by the following equalities:

$$\begin{array}{llll} 0+0=0 & 0+e=e & 0+a=a & 0+(a+e)=a+e \\ e+0=e & e+e=0 & e+a=a+e & e+(a+e)=a \\ a+0=a & a+e=a+e & a+a=0 & a+(a+e)=e \\ (a+e)+0=a+e & (a+e)+e=a & (a+e)+a=e & (a+e)+(a+e)=0 \end{array}$$

(25)

$$\begin{array}{llll} 0\cdot 0=0 & 0\cdot e=0 & 0\cdot a=0 & 0\cdot(a+e)=0 \\ e\cdot 0=0 & e\cdot e=e & e\cdot a=a & e\cdot(a+e)=a+e \\ a\cdot 0=0 & a\cdot e=a & a\cdot a=a+e & a\cdot(a+e)=e \\ (a+e)\cdot 0=0 & (a+e)\cdot e=a+e & (a+e)\cdot a=e & (a+e)\cdot(a+e)=a \end{array}$$

$$-0=0,\quad -e=e,\quad -a=a,\quad -(a+e) = a+e;$$

$$0^{-1}=0,\quad e^{-1}=e,\quad a^{-1}=a+e,\quad (a+e)^{-1}=a+e$$

It is not difficult to prove that H_Ω is equivalent to the set $\{e \neq 0\}$. Further, from the set $\Omega \cup \{a \neq 0\} \cup Ax(=)$ we obtain the following inequalities

(26) $e \neq 0,\ a \neq 0,\ a+e \neq 0,\ a \neq e,\ a+e \neq e,\ a+e \neq a.$

Finally we conclude that the set $(25) \cup (26)$ determines the required field.

We point out that a similar way can be used to construct the root field of any given equation (on some field F).

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1) i.e. model in which the sign = is interpreted as equality.

2) Each member of that set is of the following type ϕ_1, $\neg\phi_1$, $\phi_1 \wedge \dots \wedge \phi_k \implies \phi_{k+1}$, $\phi_1 \wedge \dots \wedge \phi_k \Rightarrow \neg\phi_{k+1}$ $(k=1,2,\dots)$ where ϕ_i are formulas of the form $t_1 = t_2$ $(t_1, t_2 \in Term(a,b,c,*))$.

3) We write x_1x_2, $x_1x_2x_3$, ... instead of $x_1 * x_2$, $(x_1 * x_2) * x_3$, ... respectively.

4) different as terms.

5) i.e. the equality $a = b$ can be derived from the set H by using the equational logic.

6) To speed up the algorithm we can also use every equality of the form $m_1 = m_2$ (m_1, m_2 are different markers), which is a consequence of $H_\Omega \cup Ax(=)$ (by propositional logic).

Similarly, if a formula of the form $m \neq m$ is deduced from $H_\Omega \cup Ax(=)$, then the problem $\mathcal{P}$ has no solution.

7) Other such models are the homomorphic images of the marker algebra corresponding to the congruences of the marker algebra which preserve (i.e. satisfy) the conditions $D \cup J$.

8) In any field we may use the definition $0^{-1} = 0$.

9) If a is a solution of the equation $x^2+x+e=0$, so is e+a.

R E F E R E N C E S

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ON NUCLEI AND PSEUDO-AUTOMORPHISMS OF n-ARY QUASIGROUPS

Branka P. Alimpić

In the binary case, for a quasigroup $(Q,\cdot)$ there exists the left (right) nucleus if and only if there exists at least one right (left) pseudo-automorphism [1]. In this paper we introduce i-pseudo-automorphism, $i = 1,\dots,n$, for an n-ary quasigroup, and we establish analogous connections with i-nuclei [2] of the same quasigroup.

Let (Q,ω) be an n-ary quasigroup. If a_1^{n-1} is an arbitrary element of Q^{n-1}, we shall denote it by $\overline{a}$. The bijection $L_i(\overline{a})$ of Q, $i = 1,\dots,n$, defined by $xL_i(\overline{a}) \overset{\text{def}}{=\!=} a_1^{i-1}xa_i^{n-1}\omega$, is called i-translation of (Q,ω) by the sequence $\overline{a}$.

For any mapping $\phi : Q \to Q$, $\overline{a}\phi$ denotes the sequence $a_1\phi\dots a_{n-1}\phi$.

The bijection $\lambda : Q \to Q$ with the property

$$xL_i(\overline{x})\lambda = xL_i(\overline{x}\lambda), \tag{1}$$

where $\overline{x} = x_1^{n-1}$, is called i-regular bijection of the n-ary quasigroup (Q,ω), $i = 1,\dots,n$. Let Π_i be the group of all i-regular bijections of (Q,ω). The set

$$N_i \overset{def}{=\!=} \{\bar{a} \in Q^{n-1} \mid L_i(\bar{a}) \in \Pi_i\}$$

is called i-nucleus of the n-ary quasigroup (Q,ω). The set N_i with respect to the operation o_i defined by $\bar{a} o_i \bar{b} = \bar{a} L_i(\bar{b})$, is a regular semigroup. For any n-ary quasigroup (Q,ω) there exists the nucleus N_i if and only if there exists some $\bar{e} \in Q^{n-1}$ such that

$$(\forall x \in Q)\, x L_i(\bar{e}) = x. \qquad (2)$$

In that case, the group Π_i is a homomorhic image of N_i.

Let E_i be the set of all $\bar{e} \in Q^{n-1}$ satisfying (2).

For a binary quasigroup $(Q,\cdot)$ the following equalities hold [1]:

$$N_\ell = e\mathcal{L} = \{e\lambda \mid \lambda \in \mathcal{L}\}, \quad N_r = f\mathcal{R} = \{f\rho \mid \rho \in \mathcal{R}\},$$

where e (f) is the left (right) identity, $\mathcal{L}$ ($\mathcal{R}$) is the group of left (right) regular bijections, and N_ℓ (N_r) is the left (right) nucleus of the quasigroup $(Q,\cdot)$.

In the n-ary case, we have an analogous result:

Lemma 1. $N_i = E_i \Pi_i = \{\bar{e}\lambda \mid \bar{e} \in E_i,\ \lambda \in \Pi_i\}$, $i=1,\dots,n$.

Proof. If $\bar{e}\lambda \in E_i \Pi_i$, then $x L_i(\bar{x}) L_i(\bar{e}\lambda) =$
$= x L_i(\bar{x}) L_i(\bar{e})\lambda = x L_i(\bar{x})\lambda = x L_i(\bar{x}\lambda) = x L_i(\bar{x} L_i(\bar{e})\lambda) =$
$= x L_i(\bar{x} L_i(\bar{e}\lambda))$. Hence, $\bar{e}\lambda \in N_i$.

Conversely, let $\bar{a} \in N_i$. Since (Q,ω) is an n-ary quasigroup, the equation $\bar{a} = \bar{x} L_i(\bar{a})$ in $\bar{x}$ has uniquely determined solution $\bar{e}$, which belongs to E_i. Indeed, from $x L_i(\bar{e}) L_i(\bar{a}) = x L_i(\bar{e} L_i(\bar{a})) = x L_i(\bar{a})$ we obtain $x L_i(\bar{e}) = x$, for all $x \in Q$. Thus $\bar{e} \in E_i$, and $\bar{a} = \bar{e} L_i(\bar{a}) \in E_i \Pi_i$.

Theorem 1. The i-nucleus N_i $(i=1,\dots,n)$ of an n-ary quasigroup (Q,ω) is a left group.

Proof. A semigroup is a left group if and only if it is isomorphic to the direct product of a left zero semigroup and a group. (E_i, o_i) is a left zero semigroup (for $\bar{e}, \bar{f} \in E_i$, $\bar{e} o_i \bar{f} = \bar{e} L_i(\bar{f}) = \bar{e}$), and Π_i is a group. Let us prove that $N_i \cong E_i \times \Pi_i$.

Let $\phi: E_i \times \Pi_i \to N_i$ be the mapping defined by $(\bar{e},\lambda)\phi = \bar{e}\lambda$. First we shall prove that ϕ is a bijection:

$$\begin{aligned} \bar{e}\lambda = \bar{f}\mu &\Rightarrow (\forall x \in Q)\, x L_i(\bar{e}\lambda) = x L_i(\bar{f}\mu) \\ &\Rightarrow (\forall x \in Q)\, x L_i(\bar{e})\lambda = x L_i(\bar{f})\mu \\ &\Rightarrow (\forall x \in Q)\, x\lambda = x\mu \\ &\Rightarrow \lambda = \mu. \end{aligned}$$

Since λ is a bijection, $\bar{e}\lambda = \bar{f}\lambda \Rightarrow \bar{e} = \bar{f}$. Thus,

$$\bar{e}\lambda = \bar{f}\mu \Rightarrow \lambda = \mu \wedge \bar{e} = \bar{f} \Rightarrow (\bar{e},\lambda) = (\bar{f},\mu).$$

Since $N_i = E_i \Pi_i$, ϕ is a bijection of $E_i \times \Pi_i$ onto N_i.

Finally we obtain

$$((\bar{e},\lambda)\cdot(\bar{f},\mu))\phi = (\bar{e} o_i \bar{f}, \lambda\mu)\phi = (\bar{e},\lambda\mu)\phi =$$
$$= \bar{e}\lambda\mu = \bar{e}\lambda L_i(\bar{f})\mu = \bar{e}\lambda L_i(\bar{f}\mu) = \bar{e}\lambda o_i \bar{f}\mu = (\bar{e},\lambda)\phi o_i (\bar{f},\mu)\phi.$$

Thus, ϕ is an isomorphism, as required.

Pseudo-automorphisms of n-ary quasigroups. Let (Q,ω) be a binary quasigroup. A bijection ϕ of the set Q is called a right (left) pseudo-automorphism of $(Q,\cdot)$ provided there exists at least one element a of Q, called a companion of ϕ, such that $(\phi\lambda_a, \phi, \phi\lambda_a)$ $[(\phi, \phi\rho_a, \phi\rho_a)]$ is an autotopism of $(Q,\cdot)$.

The set of all companions of a left (right) pseudo-automorphism is $a\mathcal{R}$ ($a\mathcal{L}$), where a is an arbitrary companion of ϕ, $\mathcal{R}$ ($\mathcal{L}$) is the set of all right (left) regular bijections of $(Q,\cdot)$ [1]. We prove an analogous result for n-ary quasigroups.

Let (Q,ω) be an n-ary quasigroup, $\bar{a} \in Q^{n-1}$, and let $\phi: Q \to Q$ be a bijection of the set Q.

The mapping ϕ is called i-pseudo-automorphism $(i = 1,\dots,n)$ with a companion $\bar{a}$ proveded

$$xL_i(\bar{x})\phi L_i(\bar{a}) = x\phi L_i(\bar{x}\phi L_i(\bar{a})). \qquad (3)$$

In other words, ϕ is an i-pseudo-automorphism with a companion $\bar{a}$ if and only if (α_1^{n+1}) is an autotopism of (Q,ω), where $\alpha_k = \phi L_i(\bar{a})$, $k = 1,\dots i-1, i+1,\dots,n$, $\alpha_i = \phi$. We denote this autotopism by $(\phi,\bar{a})_i$.

For n=2, we obtain the left and right pseudo-automorphisms.

Lemma 2. For any n-ary quasigroup (Q,ω) there exists at least one i-pseudo-automorphism if and only if $E_i \neq \emptyset$.

Proof. If $\bar{e} \in E_i$, the identity mapping I of Q (and every automorphism of (Q,ω), too) is an i-pseudo-automorphism with a companion $\bar{e}$.

Conversely, let ϕ be an i-pseudo-automorphism with a companion $\bar{a}$. The equation $\bar{a} = \bar{x}\phi L_i(\bar{a})$ has uniquely determined solution $\bar{e}$. We prove that $\bar{e} \in E_i$. From (3) we have

$$xL_i(\bar{e})\phi L_i(\bar{a}) = x\phi L_i(\bar{e}\phi L_i(\bar{a})) = x\phi L_i(\bar{a})$$

$$\Rightarrow xL_i(\bar{e}) = x \Rightarrow \bar{e} \in E_i.$$

COROLLARY 1. For any n-ary quasigroup (Q,ω) there exists at least one i-pseudo-automorphism if and only if $N_i \neq \emptyset$.

Let P_i be the set of all i-pseudo-automorphisms of an n-ary quasigroup (Q,ω).

Lemma 3. If $\bar{a}$ is a companion of an i-pseudo-automorphism ϕ, and $\lambda \in \Pi_i$, then $\bar{a}\lambda$ is a companion of ϕ, too.

Proof.

$$\begin{aligned} xL_i(\bar{x})\phi L_i(\bar{a}\lambda) &= xL_i(\bar{x})\phi L_i(\bar{a})\lambda && (\text{since } \lambda \in \Pi_i) \\ &= x\phi L_i(\bar{x}\phi L_i(\bar{a}))\lambda && (\text{since } \phi \in P_i) \\ &= x\phi L_i(\bar{x}\phi L_i(\bar{a})\lambda) && (\text{since } \lambda \in \Pi_i) \\ &= x\phi L_i(\bar{x}\phi L_i(\bar{a}\lambda)) && (\text{since } \lambda \in \Pi_i). \end{aligned}$$

Lemma 4. If $\bar{a}$ is a companion of an i-pseudo-automorphism ϕ, then $L_i(\bar{e}\phi o_i \bar{a}) = L_i(\bar{a})$, for every $\bar{e} \in E_i$.

Proof. Since $x\phi L_i(\bar{e}\phi o_i \bar{a}) = x\phi L_i(\bar{e}\phi L_i(\bar{a})) = xL_i(\bar{e})\phi L_i(\bar{a}) = x\phi L_i(\bar{a})$, for every $x \in Q$, we obtain $L_i(\bar{e}\phi o_i \bar{a}) = L_i(\bar{a})$.

Lemma 5. The set P_i, with respect to the composition of mappings, is a group.

Proof. Let $\phi,\psi \in P_i$, with companions $\bar{a}$ und $\bar{b}$, respectively. Since

$$x\phi L_i(\bar{a})\psi L_i(\bar{b}) = x\phi\psi L_i(\bar{a}\psi L_i(\bar{b})) = x\phi\psi L_i(\bar{a}\psi o_i \bar{b}),$$

the product of autotopisms $(\phi,\bar{a})_i$ and $(\psi,\bar{b})_i$ is the autotopism $(\phi\psi, \bar{a}\psi o_i \bar{b})_i$. Thus, $\phi\psi \in P_i$ with a companion $\bar{a}\psi o_i \bar{b}$.

The identity mapping I belongs to P_i with a companion $\bar{e} \in E_i$.

Let $\phi \in P_i$, with a companion $\bar{a}$. We prove that $\phi^{-1} \in P_i$, with a companion $\bar{e}L_i(\bar{a})^{-1}\phi^{-1}$, $\bar{e} \in E_i$. Since $(\phi,\bar{a})_i$ is an autotopism, it follows that $(\phi, a)_i^{-1}$ is an autotopism, too. Thus, we obtain

$$xL_i(\bar{x})L_i(\bar{a})^{-1}\phi^{-1} = x\phi^{-1}L_i(\bar{x}L_i(\bar{a})^{-1}\phi^{-1}).$$

and, for $\overline{x} = \overline{e}$ $(\overline{e} \in E_i)$, we have

$$xL_i(\overline{a})^{-1}\phi^{-1} = x\phi^{-1}L_i(\overline{e}L_i(\overline{a})^{-1}\phi^{-1}).$$

Hence, P_i is a group.

Let $\mathcal{A}$ be the group of all automorphisms of (Q,ω). If $\alpha \in \mathcal{A}$, and $\overline{a} \in N_i$, then $\alpha \in P_i$ with a companion $\overline{a}$. Thus, $\mathcal{A}$ is a subgroup of P_i.

Further, if $\phi \in P_i$ with a companion $\overline{a}$ and $\alpha \in \mathcal{A}$, then $\alpha\phi \in P_i$ with the same companion $\overline{a}$. Hence, every i-pseudo-automorphism ψ of the coset $\mathcal{A}\phi$ has the same companion $\overline{a}$.

THEOREM 2. Let $K_i(\phi)$ be the set of all companions of an i-pseudo-automorphism ϕ of an n-ary quasigroup (Q,ω). Then

$$K_i(\phi) = C_{\overline{a}}\Pi_i, \quad \text{where} \quad C_{\overline{a}} = \{\overline{b} \mid L_i(\overline{a}) = L_i(\overline{b})\}.$$

Proof. Let $\overline{a} \in K_i(\phi)$. By Lemma 3, we have

$$C_{\overline{a}}\Pi_i = \{\overline{b}\lambda \mid \overline{b} \in C_{\overline{a}}, \ \lambda \in \Pi_i\} \subseteq K_i(\phi).$$

Conversely, if $\overline{b} \in K_i(\phi)$, then $L_i^{-1}(\overline{b})L_i(\overline{a}) \in \Pi_i$, and

$$L_i(\overline{b}L_i^{-1}(\overline{b})L_i(\overline{a})) = L_i(\overline{b})L_i^{-1}(\overline{b})L_i(\overline{a}) = L_i(\overline{a}).$$

Hence,

$$\overline{b}L_i^{-1}(\overline{b})L_i(\overline{a}) = \overline{b}\lambda \in C_{\overline{a}}, \quad \text{and} \quad \overline{b} = (\overline{b}\lambda)\lambda^{-1} \in C_{\overline{a}}\Pi_i.$$

COROLLARY 2. $\phi \in \mathcal{A} \Rightarrow K_i(\phi) = C_{\overline{e}}\Pi_i = E_i\Pi_i = N_i$.

Lemma 6. If $\overline{a} \in K_i(\phi)$, then $C_{\overline{a}} = E_i\phi o_i\overline{a}$, and $\text{card}C_{\overline{a}} = \text{card}E_i$.

Proof. By lemma 4, $E_i\phi o_i\overline{a} \subseteq C_{\overline{a}}$. Conversely, if $\overline{b} \in C_{\overline{a}}$, let $\overline{e}$ be the solution of the equation $\overline{x}\phi o_i\overline{a} = \overline{b}$. We prove that $\overline{e} \in E_i$. Since

$$xL_i(\overline{e})\phi L_i(\overline{a}) = x\phi L_i(\overline{e}\phi L_i(\overline{a})) = x\phi L_i(\overline{b}) = x\phi L_i(\overline{a}),$$

we have $xL_i(\overline{e}) = x$. Thus $\overline{e} \in E_i$ and $C_{\overline{a}} \subseteq E_i\phi o_i\overline{a}$.

Let us consider the mapping $\Phi: E_i \to C_{\overline{a}}$ defined by $\overline{e} \mapsto \overline{e}\phi o_i\overline{a}$. Since $\overline{e}\phi o_i\overline{a} = \overline{f}\phi o_i\overline{a} \Rightarrow \overline{e}\phi = \overline{f}\phi \Rightarrow \overline{e} = \overline{f}$, for every $\overline{e},\overline{f} \in E_i$, Φ is a bijection of E_i onto $C_{\overline{a}}$. Thus, $\text{card}E_i = \text{card}C_{\overline{a}}$.

COROLLARY 3. Let (Q,ω) be an n-ary quasigroup. If $E_i \neq \emptyset$, then $\text{card}N_i = \text{card}K_i(\phi) = \text{card}E_i \cdot \text{card}\Pi_i$.

Proof. By theorem 1, we have immediately $\text{card}N_i = \text{card}E_i \cdot \text{card}\Pi_i$. Since

$$\overline{c}\lambda = \overline{b}\mu \Rightarrow L_i(\overline{c})\lambda = L_i(\overline{b})\mu \Rightarrow \lambda = \mu \wedge \overline{c} = \overline{b},$$

for every $\overline{c},\overline{b} \in C_{\overline{a}}$ and $\lambda,\mu \in \Pi_i$, the mapping $C_{\overline{a}} \times \Pi_i \to K_i(\phi) = C_{\overline{a}}\Pi_i$ defined by $(\overline{b},\lambda) \mapsto \overline{b}\lambda$ is a bijection. Thus,

$$\text{card}E_i \cdot \text{card}\Pi_i = \text{card}C_a \cdot \text{card}\Pi_i = \text{card}K_i(\phi).$$

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SOME RESULTS ON THE EXTENDING OF MODELS

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In this paper we consider so called $\wedge$-extension of models which is a generalization of the usual notion of extension in the following sense. The model $\mathcal{M}_2$ is a <u>$\wedge$-extension</u> of the model $\mathcal{M}_1$ ($\mathcal{M}_1$ and $\mathcal{M}_2$ are in the languages L_1, L_2 respectively) iff the domain of $\mathcal{M}_1$ is a subset of the domain of $\mathcal{M}_2$ and the operations and relations of $\mathcal{M}_1$ are represented (by means of the mapping $\wedge$) by terms and formulae of $\mathcal{M}_2$. The precise conditions for the mapping $\wedge$ are given in Definition 1. We prove some general results which parallel and are deeply connected with the well known results due to Łoś [6] and Malcev [7].

We start with some known definitions and assertions which we use in what follows. First of all we note that each first order language L is of the form $\mathcal{O} \cup R$ where $\mathcal{O}$ and R are the sets of operation and relation symbols of L. As usual, we denote by $\mathcal{O}(n)$ $(n=0,1,\ldots)$, $R(n)$ $(n=1,2,\ldots)$ respectively the sets of all n-ary operation and relation symbols of L. If $\mathcal{M}$ is a model (i.e. realization) of L and $f \in \mathcal{O}(n)$, $\rho \in R(n)$, then M is the domain of $\mathcal{M}$ and f_M, ρ_M are the operation and relation of $\mathcal{M}$ corresponding to f, ρ respectively. If X is the set of variables, for example

$$X = \{x_1, x_2, x_3, \ldots\},$$

then by Term(X,L) we denote the set of all terms built up from the operation symbols of L and variables of X. If t is a term built up from n different variables, $x_1, \ldots, x_n$ say, then we say that t is of length[1] n and we write $t(x_1, \ldots, x_n)$. Similarly if F is a formula in the language L having n free variables, $x_1, \ldots, x_n$ say, then n is called the length of F and F is denoted by $F(x_1, \ldots, x_n)$.

Further, For(X,L) is the set of all formulae in L built up from the variables of X. In what follows we use the symbols $u_1, u_2, \ldots$ as meta-variables.

Definition 1. Let L_1, L_2 be first order languages, $\mathcal{M}_1$, $\mathcal{M}_2$ their models and

$$\wedge : L_1 \longrightarrow \mathrm{Term}(X, L_2) \cup \mathrm{For}(X, L_2) \qquad (1)$$

a mapping which assignes to each operation symbol $f \in L_1$ a term $\hat{f} \in \mathrm{Term}(X, L_2)$ and to each relation symbol $\rho \in L_1$ a formula $\hat{\rho} \in \mathrm{For}(X, L_2)$ providing that $\hat{f}, \hat{\rho}$ are of the same lengths as f, ρ respectively[2]. We say that $\mathcal{M}_1$ is a ∧-submodel of $\mathcal{M}_2$ (and $\mathcal{M}_2$ is a ∧-extension of $\mathcal{M}_1$) iff the following conditions are satisfied:

(i) $M_1 \subseteq M_2$ (ii) $C_{M_1} = \hat{C}_{M_2}$

(iii) $(\forall u_1, \ldots, u_m \in M_1)\ f_{M_1}(u_1, \ldots, u_m) = \hat{f}_{M_2}(u_1, \ldots, u_m)$

(iv) $(\forall u_1, \ldots, u_n \in M_1)\ \rho_{M_1}(u_1, \ldots, u_n) \Leftrightarrow \hat{\rho}_{M_2}(u_1, \ldots, u_n)$

1) If t contains no variable it is of length zero.

2) A mapping of this type is called a term-formula representation (or TF-representation) of the language L_1 in the language L_2

for each $C \in \sigma_1(0)$, $f \in \sigma_1(m)$, $\rho \in R_1(n)$ $(m,n=1,2,\ldots)$, where $\hat{C}_{M_2}$, $\hat{f}_{M_2}$, $\hat{\rho}_{M_2}$ are the constant, operation and relation of $\mathcal{M}_2$ defined by the terms $\hat{C}$, $\hat{f}$ (of the length 0, m respectively) and the formula $\hat{\rho}$ (of the length n).

Using the preceding definition it is not difficult to generalize the notion of isomorphical embedding in the following way.

Definition 2. Let L_1, L_2 be first order languages, $\mathcal{M}_1$, $\mathcal{M}_2$ their models and ∧ a mapping of the form (1) satisfying the condition that the corresponding operation symbol f and term $\hat{f}$ as well as the corresponding relation symbol α and formula $\hat{\alpha}$ are of the same lengths. We say that $\mathcal{M}_1$ is ∧-embeddable in $\mathcal{M}_2$ iff there is an injection $\phi : M_1 \to M_2$ such that the following conditions are satisfied:

(i) $\phi C_{M_1} = \hat{C}_M$

(ii) $(\forall u_1, \ldots, u_m \in M_1)$

$$\phi f_{M_1}(u_1, \ldots, u_m) = \hat{f}_{M_2}(\phi u_1, \ldots, \phi u_m)$$

(iii) $(\forall u_1, \ldots, u_n \in M_1)$

$$\rho_{M_1}(u_1, \ldots, u_n) \Longleftrightarrow \hat{\rho}_{M_2}(\phi u_1, \ldots, \phi u_n)$$

for each $C \in \sigma_1(0)$, $f \in \sigma_1(m)$, $\rho \in R_1(n)$ $(n,m=1,2,\ldots)$, where $\hat{C}_{M_2}$, $\hat{f}_{M_2}$, $\hat{\rho}_{M_2}$ are the constant, operation and relation of M_2 defined by the terms $\hat{C}$, $\hat{f}$ (of the length 0, m- respectively) and the formula $\hat{\rho}$ (of the length n).

We emphasize that the notion of ∧-submodel (and accordingly ∧-embedding) is closely connected with the usual notion of submodel in the way we shall describe.

Definition 3. Let L be a first order language and D a set of explicit definitions of the form:

$D(C) \qquad C = t_1,$

$D(f) \qquad f(u_1,\dots,u_m) = t_2(u_1,\dots,u_m),$

$D(\rho) \qquad \rho(u_1,\dots,u_n) \Leftrightarrow F(u_1,\dots,u_n),$

where t_1, t_2 are terms in L of length 0, m respectively, F is a formula in L of length n and C, f, ρ are new symbols for L, i.e. they do not belong to L. The language L_D obtained by adding to L all new operation and relation symbols defined by the explicit definitions of D is called a D-extension of L.

Using well known theorems about explicit definitions [6] it is easy to prove the following theorem.

Theorem 1. Let L be a first order language, D a set of explicit definitions of the form D(C), D(f), D(ρ) and L_D corresponding D-extension of L. Each model $\mathcal{M}$ of L can be extended, in the unique way, to a model $\mathcal{M}_D$ of L_D in which all the definitions of D are satisfied, and conversely each model $\mathcal{M}_D$ of L_D in which the definitions of D are satisfied can be restricted, in the unique way, to a model $\mathcal{M}$ of L. Note that the domain M_D equals M.

The following theorem links the notions $\wedge$-submodel and D-extension.

Theorem 2. Let L_1, L_2 be first order languages, $\mathcal{M}_1$, $\mathcal{M}_2$ their models and suppose that $\mathcal{M}_1$ is a $\wedge$-submodel of $\mathcal{M}_2$ where $\wedge$ is a mapping of the form (1) satisfying the conditions of Definition 1.

Then there is a set D of explicit definitions of the form D(C), D(f), D(ρ) such that $\mathcal{M}_1$ is a submodel of $\mathcal{M}_{2D}$.

Conversely, let L_2 be a first order language, D a set of explicit definitions in L_2 and $\mathcal{M}_{2D}$ D-extension of $\mathcal{M}_2$. If the model $\mathcal{M}_1$ of the language L_1 is a submodel of $\mathcal{M}_{2D}$, then there is a mapping

$$\wedge : L_1 \to Term(X,L_2) \cup For(X,L_2)$$

satisfying the conditions of Definition 1 such that $\mathcal{M}_1$ is a $\wedge$-submodel of M_2.

Hint. If $\mathcal{M}_1$ is a $\wedge$-submodel of $\mathcal{M}_2$, then D consists of definitions of the following form:

$$C = \hat{C}, \quad f(u_1,\dots,u_m) = \hat{f}(u_1,\dots,u_m),$$
$$\rho(u_1,\dots,u_n) \Leftrightarrow \hat{\rho}(u_1,\dots,u_n)$$

Conversely, if $\mathcal{M}_1$ is a submodel of $\mathcal{M}_{2D}$, then the unique mapping $\wedge$ is the following:

$$\wedge = \begin{pmatrix} C & f & \rho \\ t_1 & t_2 & F \end{pmatrix}$$

$$(C \in \mathcal{O}_1(0),\ f \in \mathcal{O}_1(m),\ \rho \in R_1(n)),$$

where the formulae

$$C = t_1, \quad f(u_1,\dots,u_m) = t_2(u_1,\dots,u_m),$$
$$\rho(u_1,\dots,u_n) \Leftrightarrow F(u_1,\dots,u_n)$$

are elements of D.

Definition 4. Let L_1, L_2 be first order languages, D a set of explicit definitions in L_2, L_{2D} the corresponding D-extension of L_2 and $L_1 \subseteq L_{2D}$. Further, let F_2 be a formula in L_2.

We say that F_2 is expessible in the language L_1 iff there is a formula F_1 in L_1 such that

$$D \vdash F_1 \Leftrightarrow F_2 \qquad (2)$$

For example, if $L_1 = \{f,\rho\}$, $L_2 = \{*,\alpha\}$, $D = \{f(x) = x*x,$ $\rho(x) \Longleftrightarrow (\forall y).\ \alpha(x,y)\}$, then the formula

$$(\exists x)(\forall y)\alpha(x*x,y) \Longrightarrow (\forall y)\alpha((x*x)*(x*x),y)$$

is expressible in L_1 by

$$(\exists x)\ \rho\ (fx) \Longrightarrow \rho(\iota^{-}x)$$

Theorem 3. Let L_1, L_2 be first order languages, L_{2D} a D-extension of L_2 and $L_1 \subseteq L_{2D}$. Further let $\mathcal{M}_1$, $\mathcal{M}_2$ be models of L_1, L_2 such that $\mathcal{M}_1$ is a submodel of $\mathcal{M}_{2D}$. Then for each formula F_2 in L_2 expressible by a universal formula F_1 in L_1 the following implication

$$\mathcal{M}_2 \models F_2 \rightarrow \mathcal{M}_1 \models F_1 \qquad (3)$$

holds.

Proof. Suppose that

$$\mathcal{M}_2 \models F_2 \qquad (4)$$

holds. By assumption of the theorem we have

$$D \vdash F_1 \Longleftrightarrow F_2$$

wherefrom it follows immediately

$$\mathcal{M}_{2D} \models F_1 \quad \text{iff} \quad \mathcal{M}_{2D} \models F_2.$$

Since F_2 is in L_2 we conclude

$$\mathcal{M}_{2D} \models F_2 \quad \text{iff} \quad \mathcal{M}_2 \models F_2.$$

From the preceding two equivalences it follows that

$$\mathcal{M}_2 \models F_2 \quad \text{iff} \quad \mathcal{M}_{2D} \models F_1$$

and using the assumption (4) we obtain

$$\mathcal{M}_{2D} \models F_1.$$

Since $\mathcal{M}_1$ is a submodel of $\mathcal{M}_{2D}$ and F_1 is a universal formula, using the well known theorem [6] we deduce

$$\mathcal{M}_1 \models F_2$$

which completes the proof.

Using the notion of $\wedge$-submodel, Definition 4 and Theorem 3 may be reformulated in the following way.

Definition 4´. Let L_1, L_2 be first order languages, $\wedge$ a mapping of the form (1) and F_2 a formula in L_2. We say that F_2 is expressible in the language L_1 iff there is a formula F_1 in L_1 such that

$$\hat{F}_1 = F_2. \qquad (5)$$

Theorem 3´. Let L_1, L_2 be first order languages, $\wedge$ a mapping of the form (1), i.e. $\wedge$ is a TF-representation of L_1 in L_2 and $\mathcal{M}_1$, $\mathcal{M}_2$ models of L_1, L_2 such that $\mathcal{M}_1$ is a $\wedge$-submodel of $\mathcal{M}_2$. Then for each formula F_2 in L_2 which is expressible in L_1 (in the sence of the preceding definition) by universal formula F_1, the implication (3) holds.

Using the notion of diagram the following theorem can be easily proved.

Theorem 4. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be models of the languages L_1, L_2, $\wedge$ a TF-representation of L_1 in L_2 and $D(\mathcal{M}_1)$ a diagram of $\mathcal{M}_1$ obtained by adding new constant symbols to L_2 - the names for elements of M_1. Then $\mathcal{M}_1$ is a $\wedge$-submodel of $\mathcal{M}_2$ iff $\mathcal{M}_2$ is a model of $D(\mathcal{M}_1)$.

We now prove the theorem which gives necessary and sufficient conditions for $\wedge$-extension of a given model to some model of a given set of formulae. The theorem parallels to the corresponding theorem due to Łoś [6].

Theorem 5. Let L_1, L_2 be first order languages, $\mathcal{M}_1$ a model of L_1, $\wedge$ a TF-representation of L_1 in L_2

and F a set of formulae in L_2. $\mathcal{M}_1$ can be $\wedge$-extended[3] to some model of $\mathcal{F}$ iff for each formula A_2 in L_2 which is expressible by a universal formula A_1 in L_1 the following condition

$$\mathcal{F} \vdash A_2 \rightarrow \mathcal{M}_1 \models A_1 \tag{6}$$

holds.

Proof. The condition (6) is necessary, for if $\mathcal{M}_2$ is a $\wedge$-extension of $\mathcal{M}_1$ which is a model for $\mathcal{F}$ that $\mathcal{M}_2$ is a model for each consequence A_2 of $\mathcal{F}$. If A_2 is expressible by a universal formula A_1 in L_1 then by theorem 3´ the following implication

$$\mathcal{M}_2 \models A_2 \rightarrow \mathcal{M}_1 \models A_1 \tag{7}$$

holds. Further, using the assumption $\mathcal{M}_2 \models F$ we obtain

$$\mathcal{F} \vdash A_2 \rightarrow \mathcal{M}_2 \models A_2 \tag{8}$$

From (7), (8) we conclude immediately (6). Conversely, suppose that for each formula A_2 in L_2 expressible by a universal formula A_1 in L_1 the condition (6) is satisfied and that $\mathcal{M}_1$ cannot be $\wedge$-extended to any model of $\mathcal{F}$. Using the preceding theorem the latter assumption yields that the set $D(\mathcal{M}_1) \cup \mathcal{F}$ has no model. By the compactness theorem it follows that for some finite subset K of $D(\mathcal{M}_1)$ the set $\hat{K} \cup \mathcal{F}$ has no model. Let $a_1, \ldots, a_n$ be all names of elements of M_1 appearing in K and let

$$F(a_1, \ldots, a_n)$$

be the conjunction of all formulae of K. The formula F being the conjunction of some formulae of $D(\mathcal{M}_1)$ contains no variables and no quantifiers. Further, if G is

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[3] i.e. there exists a model $\mathcal{M}_2$ of $\mathcal{F}$ which is a $\wedge$-extension of $\mathcal{M}_1$.

the conjunction of all formulae of $\hat{K}$, then it is easy to verify that G is just $\hat{F}$. Since $\hat{K} \cup \mathcal{F}$ has no model we conclude that neither has the set $\{\hat{F}\} \cup \mathcal{F}$, wherefrom it follows immediately

$$\mathcal{F} \vdash \neg \hat{F}(a_1, \ldots, a_n). \tag{9}$$

As the constants $a_1, \ldots, a_n$ are new for L_2, (9) yields

$$\mathcal{F} \vdash (\forall x_1, \ldots, x_n) \neg \hat{F}(x_1, \ldots, x_n). \tag{10}$$

Let us consider the formula

$$(\forall x_1, \ldots, x_n) \neg \hat{F}(x_1, \ldots, x_n). \tag{11}$$

Obviously it is in the language L_2 and is expressible in L_1 by the universal formula

$$(\forall x_1, \ldots, x_n) \neg F(x_1, \ldots, x_n). \tag{12}$$

Since $F(a_1, \ldots, a_n)$ is the conjunction of some formulae of $D(\mathcal{M}_1)$, it follows immediately that

$$\mathcal{M}_1 \models F(a_1, \ldots, a_n), \tag{13}$$

wherefrom we conclude:

$$\text{Not } \mathcal{M}_1 \models (x_1, \ldots, x_n) \neg F(x_1, \ldots, x_n). \tag{14}$$

(10) and (14) contradicts to the assumption (6) and the proof of our theorem is completed.

By means of theorem 5 it can be proved the following theorem which has a great number of applications.

Theorem 6. Let L_1, L_2 be first order languages, $\mathcal{F}$ a set og formulae in L_2 and $\wedge$ a TF-representation of L_1 in L_2. Every model $\mathcal{M}_1$ of L_1 can be $\wedge$-extended to some model $\mathcal{M}_2$ of the set $\mathcal{F}$ iff for every consequence A of $\mathcal{F}$ which is expressible in L_1 by a universal formula A´ the following condition

$$\text{F´ is valid} \tag{15}$$

holds[4)].

In the following we list some applications of the preceding theorem.

1. Every model of the language $\{\rho\}$, where ρ is a unary relation symbol, can be $\wedge$-extended to a model of the formula

$$(\forall x,y)(\alpha(x,y) \Rightarrow \alpha(y,x)) \qquad (16)$$

It suffices to represent ρ by the formula $\alpha(x,x)$.

2. Every model of the language R, where R contains only unary relation symbols can be $\wedge$-extended to a model of the formula

$$(\forall x,y,z)(\alpha(x,y,z) \Rightarrow \alpha(z,y,x)). \qquad (17)$$

Namely, if $\rho \in R$, then for $\hat{\rho}$ we choose the formula

$$\alpha(x,\bar{\rho}, x),$$

where $\bar{\rho}$ is a new constant symbol.

3. Every model of any relational language R can be $\wedge$-extended to a model of the formula

$$(\forall x,y)(\alpha(x*y) \wedge \alpha(y*x) \Rightarrow x = y) \qquad (18)$$

We note that in this case the relation symbol $\rho \in R(n)$ is represented by the formula $\alpha(t*t)$, where t is a term of the form $(\dots((\bar{\rho}*x_1)*x_2) \dots x_n)$ ($\bar{\rho}$ is a new constant symbol corresponding to ρ).

4) We note that in the case when L_1 contains the equality symbol =, the notion F′ is valid is defined as follows: F′ is satisfied in all normal models of L_1, i.e. in all models where = is interpreted as equality.

4. Every model of any first order language L can be $\wedge$-extended to some model of the set $\mathcal{F}$:

$$(\forall x,y)\ (\alpha(x,y) \Rightarrow \alpha(y,x)$$
$$(\forall x,y,z)\ (x*y)*z = x*(y*z). \qquad (19)$$

Hint. First we define the binary operation o as follows:

$$xoy = (a*x)*y \text{ (a is a new constant symbol)}$$

Further, if $f \in \sigma(n)$, then $\hat{f}$ is the term

$$(\dots((\bar{f} \circ x_1) \circ x_2) \dots \circ x_n)$$

($\bar{f}$ is a new constant symbol corresponding to f)

and if $\rho \in R(n)$, then $\bar{\rho}$ is the formula

$$\alpha(t,t),$$

where t is of the form

$$(\dots((\bar{\rho} \circ x_1) \circ x_2) \dots \circ x_n)$$

($\bar{\rho}$ is a new constant symbol corresponding to ρ).

Remark. In his doctoral thesis "Subalgebras of groupoids" S.Markovski has recently obtained several results for Ω-algebras which are very closed to the results of this paper.

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TWO REMARKS ON BOOLEAN ALGEBRAS

Žarko Mijajlović

1. Boolean formulas and modal logic. Let $L_{BA} = \{+,\cdot,{}',0,1\}$ denote the language of Boolean algebras (shortly BA) with the natural interpretations of symbols of L_{BA}, $Term_{L_{BA}}$ the set of Boolean terms, and $For_{L_{BA}}$ the set of quantifier-free formulas of L_{BA}. Further, let $Sent_{S5}$ denote the set of all modal propositional sentences, where L denotes the modal necessity operator, and M the modal possibility operator. The subscript S5 is used to denote the modal logic S5 which has, besides axioms of propositional calcalus, also the following axioms for modal operators:

$$L\phi \Longrightarrow \phi, \quad L(\phi \Longrightarrow \psi) \Longrightarrow (L\phi \Longrightarrow L\psi), \; M\phi \Longrightarrow LM\phi.$$

The inference rules are Modus Ponens and the necessitation rule: $\vdash \phi$ implies $\vdash L\phi$.

We show that a map

$$\sigma : Term_{L_{BA}} \cup For_{L_{BA}} \longrightarrow Sen_{S5}$$

can be defined in a natural way. To define σ let $\phi \in For_{L_{BA}}$. Then the **modal transform** of ϕ, $\sigma\phi$, is defined as follows:

- All occurences of Boolean operation symbols

+,.,´,0,1 in ϕ are replaced by logical symbols $\vee, \wedge, \neg$, "false", "true", respectively.

-Each occurence of the equality sign = in ϕ is replaced by strict equivalence $\equiv$, and the sign $\leq$ by the strict implication $\prec$.

Then the following theorem holds:

Theorem 1.1. Let $\phi \in For_{L_{BA}}$. Then ϕ is true in all Boolean algebras iff $\sigma\phi$ is a theorem of modal logic S5, i.e.

$$BA \models \phi \quad \text{iff} \quad \vdash_{S5} \sigma\phi.$$

Remark 1.2. In above, if u,v are Boolean terms, then the subformula u=1 of ϕ can be replaced by $L\sigma u$, and $v > 0$ by $M\sigma v$.

Example 1.3. Let $\phi \in For_{L_{BA}}$ be $(p=1) \wedge (q' \leq p') \Leftrightarrow \Leftrightarrow (p \cdot q=1)$. Then $\sigma\phi$ is $Lp \wedge (\neg q \prec \neg p) \Leftrightarrow L(p \wedge q)$. As $BA \models \phi$ we have $\vdash_{S5} \sigma\phi$.

It should be remarked that an appropriate logic for quantifier-free formulas, call it Σ_0-logic, can be designed. To be more precise, the quantifier-free fragment of predicate calculus can be axiomatized taking all instances of tautologies for For_L, then axioms for the equality sign =, and two rules of inferences: Modus Ponens, and the substitution rule $\frac{\phi(x)}{\phi(w)}$, $w \in Term_L$.

The completeness theorem holds for any set $T \subseteq For_L$:

Theorem 1.4. $T \vdash_{\Sigma_0} \phi$ iff $T \models \phi$.

Here, $T \vdash_{\Sigma_0} \phi$ denotes that ϕ has a deduction from T in Σ_0-logic, and $T \models \phi$ means that (the universal closure of) ϕ holds in all models of T.

As a consequence of Theorem 1.4. we have the following. Let T_{BA} be the theory of Boolean algebras. Then T_{BA} may be considered as a formalization of the theory of first-degree modal sentences of S5, what probably may add a new feature to a semantical theory of modal logics.

Proofs of the previous theorems will be published somewhere else.

2. Horn Boolean sentences. It is well-known the following theorem:

Theorem 2.1. (Vaught). Let ϕ be a Horn sentence [1)] in the language L_{BA}. If $\underline{2} \models \phi$, then $BA \models \phi$.

Here $\underline{2}$ denotes two-element Boolean algebra.

It was first S.Prešić (1976, private communication) who observed that procedures of solving, and forms of solutions of certain Boolean equations can be transfered from two-element Boolean algebra to any Boolean algebra. These ideas are developed in [4], where it is shown how Vaught´s theorem makes the greatest part of the teory of Boolean equations trivial. We give some new evidences for that. First, we give an elementary proof of Vaughts theorem.

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1) Horn formulas over a language L are defined as follows:

-Elementary Horn formulas are defined as all atomic formulas of L, and all formulas of the form $\psi_1 \wedge \ldots \wedge \psi_n \Rightarrow \psi$, where $\psi_1, \ldots, \psi_n, \psi$ are atomic.

- Every Horn formula is built from elementary Horn formulas by use of $\wedge, \forall, \exists$.

Proof of Theorem 2.1. First, let ϕ be an elementary Horn formula, $\phi \in For_{L_{BA}}$. Assume $\underline{2} \models \phi$. Then, by a Horn's theorem, ϕ holds on all Boolean algebras 2^n, $n \in \{1,2,\ldots\}$. As every finite BA is a power of $\underline{2}$, it follows that ϕ holds on all finitely generated BA's. ϕ is universal, hence $BA \models \phi$.

Now, let ϕ be any Horn sentence. Then ϕ is equivalent to a formula θ in prenex normal form, where the matrix $\psi(x_1,\ldots,x_n)$ of θ is also a Horn formula. By use of Skolem function symbols, quantifiers in θ are eliminated, i.e. there are terms $t_1,\ldots,t_n$ built from Skolem functions so that for the Skolem expansion T^S_{BA} of T_{BA} we have $T^S_{BA} \vdash \phi \Longleftrightarrow \psi(t_1,\ldots,t_n)$. As $\underline{2} \models \phi$, for a Skolem expansion $\underline{2}^S$ of $\underline{2}$ we have $\underline{2}^S \models \psi(t_1,\ldots,t_n)$. The set of Boolean functions in 2^n is complete, i.e. every function $f:2^n \to 2$ is Boolean, hence we may assume that $t_1,\ldots,t_n$ are Boolean terms, and so $\underline{2} \models \psi(t_1,\ldots,t_n)$. But $\psi(t_1,\ldots,t_n)$ is elementary Horn formula (as it does not contain quantifiers), thus by the first part of the proof, $BA \models \psi(t_1,\ldots,t_n)$. T^S_{BA} is a conservative extension of T_{BA}, $\phi \Longleftrightarrow \psi(t_1,\ldots,t_n)$ is in the language L_{BA}, so $T_{BA} \models \phi \Longleftrightarrow \psi(t_1,\ldots,t_n)$; hence $T_{BA} \models \phi$, i.e. $BA \models \phi$.

We remark that the proof of the above theorem can be obtained by use of Theorem 1.1 and Carnap decision procedure for modal propositional calculus S5, cf. [2], p. 116, hence by use of purely syntactical transformations.

As an application we shall consider the following system of Boolean equations over a Boolean algebra B.

$$(S)\qquad \begin{array}{c} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = b_n \end{array} \qquad a_{ij}, b_i \in B,\ 1 \le i,\ j \le n.$$

Let $A = \| a_{ij} \|$ be the matrix of the system (S) and $b = (b_1,\ldots,b_n)$. We shall be interested when (S) has a solution for arbitrary $b \in B$. The following theorem gives the necessary and sufficient condition, and a description of solutions if this condition is satisfied.

Theorem 2.2. Let A be the matrix of the system (S). Then the following hold in B:

1° $\quad \forall y_1\ldots y_n \exists x_1\ldots x_n \ Ax = y \Longleftrightarrow A^T A = I$,

I is the identity matrix, A^T is the transpose of A.

2° $\quad A^T A = I \Rightarrow \forall y_1\ldots y_n \ \forall x_1\ldots x_n (Ax = y \Longleftrightarrow x_1 = D_1 \wedge \ldots \wedge x_n = D_n)$,

D_i is the determinat obtained from the matrix A by replacing i-th column by the vector $y = (y_1,\ldots,y_n)$. The determinant of matrix A is defined by

$$\det(A) = \sum_{p \in P} a_{1p_1} \cdots a_{np_n},$$

P is the set of all permutations of $\{1,2,\ldots,n\}$.

Proof. Let ϕ denotes the formula in 1°, and ψ the formula in 2°. First, we observe that both these formulas are Horn formulas, as $Ax=y$ can be represented as the conjunction of the equations of the system (S). Thus, if we prove universal closures of ϕ, ψ in the Boolean algebra $\underline{2}$, then by the Theorem 2.1, ϕ and ψ will hold in all BA's. Therefore we check 1° and 2° in $\underline{2}$.

1° ($\Rightarrow$) Assume

$$\underline{2} \models \forall y_1\ldots y_n \exists x_1\ldots x_n \ Ax=y. \tag{1}$$

Let $a_i=(a_{1i},\dots,a_{ni})$, $1 \le i \le n$. Then from (1) it follows that for each $b \in \underline{2}^n$ there exist $x_1,\dots,x_n \in \underline{2}$ so that $b = \sum_{i=1}^{n} x_i a_i$, i.e. $a_1,\dots,a_n$ span $\underline{2}^n$, so $a_1,\dots,a_n$ are atoms of $\underline{2}^n$. Hence, it may be assumed that $a_i=(0,\dots,0,1,0,\dots,0)$, 1 stands on i-th place, thus $A^T A=I$. ($\Longleftarrow$) If $A^T A=I$, then $x=A^T b$ satisfies $Ax=b$, $b \in \underline{2}^n$.

2^o Assume $\underline{2} \models A^T A=I$. Then all entries of A are 0's and 1's, and each row and each column has exactly one occurence of 1. Thus we have, also, $\underline{2} \models AA^T=I$. Let Δ denotes the symmetric difference. Then $(2,\Delta,\cdot,0,1)$ is a field, and $x+y=x\Delta y\Delta xy$. By the ortogonality of row vectors and, also, column vectors in A, it follows that (S) is equivalent to the system (S^Δ) obtained from (S) by replacing all occurences of + in (S) by Δ. If in the definition of determinant, + is replaced by Δ, we have $\det(A^T)\det(A)=1$, i.e. $\det(A)=1$, thus by Cramer's rule

$$\underline{2} \models Ax=y \iff x_1=D_1^\Delta \wedge \dots \wedge x_n=D_n^\Delta,$$

D_i^Δ is the determinant in terms of Δ.

Again by the ortogonality of row vectors and column vectors in A, the symmetric difference in D_i^Δ can be replaced by +, therefore

$$\underline{2} \models Ax=y \iff x_1=D_1 \wedge \dots \wedge x_n=D_n. \quad \dashv$$

The system (S) is also considered in [5], and the part 1^o of Theorem 2.2 is proved there, however by use of other methods.

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It is simple to prove that

LEMMA 1. (DeM) $\overline{x \vee y} = \overline{x} \wedge \overline{y}$, $\overline{x \wedge y} = \overline{x} \vee \overline{y}$ holds, vhere $\vee$ and $\wedge$ are l.u.b. and g.l.b. in $(L,\leq)$. ∎

Separate study has been given to the structures defined above. These have been called De Morgan lattices in Monteiro [1960], distributive involution lattices (i-lattices) in Kalman [1958], and quasi-Boolean algebras in Białnicki-Birula and Rasiowa [1957]. Most commonly used name is the first one. We shall use it.

De Morgan lattices were investigated in standard manner (see Rasiowa [1974]). The rise of interest in them is connected with their applications onto various logical systems. Main device are so called truth-filters.

Def. 2. A filter F in a de Morgan lattice L is

i) consistent iff $\neg(\exists x \in L)(x \in F$ and $\overline{x} \in F)$

ii) complete iff $(\forall x \in L)(x \in F$ or $\overline{x} \in F)$

iii) truth-filter iff it is complete and consistent. ∎

LEMMA 2. F is a truth-filter iff

$$(\forall x \in L)(x \in F \text{ iff } \overline{x} \notin F). \blacksquare$$

It is obvious that the truth-filters corresponds to the ultrafilters in a Boolean algebra.

Belnap and Spencer [1966] has proved that

THEOREM. 1. There exists a truth filter in a De Morgan lattice L iff (C3) $(\forall x \in L)\ \overline{x} \neq x$. ∎

We will call De Morgan lattice satisfying (C3) regular.

The theorem above states the mere fact of existence

of a truth-filter in a De Morgan lattice which is regular. It is our aim to obtain Stone-type theorems about these filters, i.e. to find necessary and sufficient conditions for existence of a truth-filter containing a given filter or a given point. To begin with, we are developing some necessary notions and appropriate algebraic apparatus.

Def. 3.

$I(A) = \{x \in L \mid (\exists a_1,\ldots,a_n \in A)\ x \leq a_1 \vee \ldots \vee a_n\}$

$F(A) = \{x \in L \mid (\exists a_1,\ldots,a_n \in A)\ x \geq a_1 \wedge \ldots \wedge a_n\}$

$T_o = \{x \in L \mid x = \overline{x} \wedge x\}$. $I_o = I(T_o)$

$T_1 = \{x \in L \mid x = \overline{x} \vee x\}$, $F_1 = F(T_1)$

$\overline{A} = \{\overline{x} \in L \mid x \in A\}$, $a \wedge A = \{a \wedge x \mid x \in A\}$

$A \wedge B = \{x \wedge y\} \mid x \in A$ and $y \in B\}$,

$[a,b] = \{x \in L \mid a \leq x \leq b\}$.

I(A) and F(A) will be called ideal and filter generated by the set A. T_o and T_1 will be called set of zeroes and set of units.

The following lemma states some usefull properties of the notions introduced above.

LEMMA 3. Let $x,y \in L$, then:

i) $x \in T_o \Leftrightarrow x \leq \overline{x}$, $x \in T_o$ and $y \leq x \Rightarrow y \in T_o$.

ii) $x \in T_1 \Leftrightarrow x \geq \overline{x}$, $x \in T_1$ and $y \geq x \Rightarrow y \in T_1$.

iii) $x \wedge \overline{x} \in T_o$, $x \vee \overline{x} \in T_1$.

iv) $\overline{T_o} = T_1$, $\overline{T_1} = T_o$.

v) F is a filter iff $\overline{F}$ is an ideal.

vi) $\overline{I_o} = F_1$, $\overline{F_1} = I_o$.

vii) $\overline{I(A)} = F(\bar{A})$, $\overline{F(A)} = I(\bar{A})$.

viii) L is regular iff $T_o \cap T_1 = \emptyset$.

ix) A filter F is consistent iff $F \cap T_o = \emptyset$.

x) If F is a truth-filter, then F is a prime filter.

xi) If F is a truth-filter, then $F \supseteq F_1$ and $F \cap I_o = \emptyset$.

Proof. Tedious, but straightforward. ■

Def. 4. Let $p \in T_1$, define $f_p : L \to L$ as $f_p(x) = p \wedge (\bar{p} \vee x)$, for $x \in L$. ●

We will call the functions introduced above p-mappings.

LEMMA 4. Let $p, q \in T_1$ and $x \in L$, then:

i) $f_p(x) = \bar{p} \vee (p \wedge x)$

ii) f_p is a homomorphism from L into L.

iii) $f_p(L) = [\bar{p}, p]$.

iv) $f_p o f_p = f_p$.

v) $f_p o f_q = f_{f_p(q)}$.

vi) $f_p = f_q \Rightarrow p = q$.

vii) $x \in T_o \Rightarrow f_p(x) \in T_o$, $x \in T_1 \Rightarrow f_p(x) \in T_1$.

viii) $x \in I_o \Rightarrow f_p(x) \in I_o$, $x \in F_1 \Rightarrow f_p(x) \in F_1$.

ix) $f_p(p \wedge x) = f_p(x)$, $f_p(p \vee x) = p$,

$f_p(\bar{p} \vee x) = f_p(x)$, $f_p(\bar{p} \wedge x) = \bar{p}$.

x) If T is a truth-filter, then $x \in T \Rightarrow f_p(x) \in T$.

Proof. Straightforward. ■

LEMMA 5. $(a \wedge T_1) \cap T_o \neq \emptyset$ iff $(\exists p \in T_1) f_p(a) \in T_o$.

Proof. Simple application of (vii) and (ix) from Lemma 4. ■

THEOREM 1. $(a \wedge T_1) \cap T_o \neq \emptyset$ iff $(a \wedge F_1) \cap T_o \neq \emptyset$.

Proof. ($\Longrightarrow$) Trivial, because of $T_1 \subseteq F_1$.

($\Longleftarrow$) Let $(a \wedge F_1) \cap T_o \neq \emptyset$, i.e. let there exists $f \in F_1$ such that $a \wedge f \in T_o$ ... (1). By the definition of F_1 we have that there exist $p_1, \ldots, p_n \in T_1$ such that $f \geq p_1 \wedge \ldots \wedge p_n$ and $a \wedge f \geq a \wedge p_1 \wedge \ldots \wedge p_n$ ... (2). Applying (i) of Lemma 3 onto (1) and (2) it follows that

$$a \wedge p_1 \wedge \ldots \wedge p_n \in T_o \qquad (3)$$

for some $p_1, \ldots, p_n \in T_1$.

Applying f_{p_n} onto $a \wedge p_1 \wedge \ldots \wedge p_n$ and using (ii), (vii), (viii) and (ix) of Lemma 4 we have:

$$f_{p_n}(a \wedge p_1 \wedge \ldots \wedge p_n) = f_{p_n}(a \wedge p_1 \wedge \ldots \wedge p_{n-1}) =$$

$$= f_{p_n}(a) \wedge f_{p_n}(p_1) \wedge \ldots \wedge f_{p_n}(p_{n-1}) =$$

$$= f_{p_n}(a) \wedge p_1^1 \wedge \ldots \wedge p_{n-1}^1 \in T_o \qquad (4)$$

where $p_1^1, \ldots, p_{n-1}^1 \in T_1$. Hence, (3) is reduced to a similar formula (4) with the number of unites reduced by one. Repeating this procedure we will obtain that for some p_1^{n-1}, $p_2^{n-2}, \ldots, p_{n-1}^1$, $p_n \in T_1$

$$f_{p_1^{n-1}}(f_{p_2^{n-2}}(\ldots(f_{p_{n-1}^1}(f_{p_n}(a)))\ldots)) \in T_o \ \ldots \ (5)$$

holds. As, by (v) and (vii) of Lemma 4, a composition of p-mappings is a p-mapping, we infer that there exists $p \in T_1$ such that $f_p(a) \in T_o$. Because of Lemma 5, the last is equivalent to $(a \wedge T_1) \cap T_o \neq \emptyset$. ■

COROLLARY 1.1. $a \in I_o$ implies $(a \wedge T_1) \cap T_o \neq \emptyset$.

Proof. $a \in I_o$ implies that $\bar{a} \in \bar{I}_o = F_1$. Hence, $a \wedge \bar{a} \in a \wedge F_1$. But, $a \wedge \bar{a} \in T_o$, so $(a \wedge F_1) \cap T_o \neq \emptyset$, and, by theorem above $(a \wedge T_1) \cap T_o \neq \emptyset$. ■

Now, we are ready to prove our main theorem.

THEOREM 2. Let F be a filter in a De Morgan lattice L. F is contained in some truth-filter T iff

$$(F \wedge T_1) \cap T_o = \emptyset.$$

Proof. ($\Longrightarrow$) Suppose the opposite - i.e. let there exists a truth-filter $T \supseteq F$ and $(F \wedge T_1) \cap T_o \neq \emptyset$ which implies $(a \wedge T_1) \cap T_o \neq \emptyset$ for some $a \in F$, which is equivalent (according to Lemma 5) with $(\exists p \in T_1) f_p(a) \in T_o$. As $a \in F \subseteq T$, we have $a \in T$, and (according to (x) of Lemma 4) $f_p(a) \in T$. Hence, $T \cap T_o \neq \emptyset$ which contradicts (xi) of Lemma 3.

($\Longleftarrow$) Suppose $(F \wedge T_1) \cap T_o = \emptyset$. We will prove the existence of the truth-filter containing F in several steps.

1^o Filter $G = F(F \cup T_1)$ is consistent.

If not, by (ix) of Lemma 3, there exists $g \in T_o$ such that $g \in G$. By definition of G, $g \geq a \wedge r$ where $a \in F$ and $r \in F(T_1) = F_1$. As $g \in T_o$, we have that $a \wedge r \in T_o$, i.e. $(a \wedge F_1) \cap T_o \neq \emptyset$ which is (according to Theorem 1) equivalent with $(a \wedge T_1) \cap T_o \neq \emptyset$ which implies $(F \wedge T_1) \cap T_o \neq \emptyset$. A contradiction.

2^o The set of all consistent filters containing G has a maximal element.

This set $\mathcal{C}$ is non-empty as $G \in \mathcal{C}$. $\mathcal{C}$ is also closed over the unions of chains (proof is simple). Hence, by Zorn's Lemma, $\mathcal{C}$ has a maximal element T.

3^o T is a truth-filter and contains F.

$F \subseteq G \subseteq T$, so, T contains F. T is consistent since it belongs to $\mathcal{C}$. It suffices to prove that T is complete.

If it is not the case, we have that for some $a \in L$, $a, \bar{a} \notin T$. Then the filter T is a proper subfilter of $F(T,a)$ and $F(T,\bar{a})$ so they don't belong to $\mathcal{C}$, hence, they are inconsistent. By (ix) of Lemma 3 $(\exists q_1, q_2 \in T_o)(q_1 \in F(T,a)$ and $q_2 \in F(T,\bar{a}))$. By their construction we have $q_1 \geq t_1 \wedge a$ and $q_2 \geq t_2 \wedge \bar{a}$ for some $t_1, t_2 \in T$. Then it follows that $q_1 \vee q_2 \geq t_1 \wedge t_2 \wedge (a \vee \bar{a})$ and as $a \vee \bar{a} \in T$ we obtain that $q_1 \vee q_2 \in T$. Hence, $T \cap I_o \neq \emptyset$ which contradicts the consistency of T because of $T \supseteq F(T_1) = F_1 = \bar{I}_o$. ■

COROLLARY 2.1. There is a truth-filter containing a iff $(a \wedge T_1) \cap T_o = \emptyset$.

Proof. Let F_a be a filter generated by {a}. There exists a truth-filter containing {a} iff there exists a truth-filter containing F_a. By Theorem 2, the last is equivalent with $(F_a \wedge T_1) \cap T_o = \emptyset$, which is, because of the definition of F_a, T_1, T_o, equivalent to $(a \wedge T_1) \cap T_o = \emptyset$. ■

COROLLARY 2.2. Let $a, b \in L$ such that $\neg a \leq b$. There exists a truth-filter T such that $a \in T$ and

$b \notin T$ iff $(a \wedge \bar{b} \wedge T_1) \cap T_o = \emptyset$.

Proof. Because of the completeness of every truth-filter, the filter with desired property fulfills $a \in T$, $\bar{b} \in T$ and $a \wedge \bar{b} \in T$. Because of Corollary 2.1 such filter exists iff $(a \wedge \bar{b} \wedge T_1) \cap T_o = \emptyset$. ■

COROLLARY 2.3. There exists a truth-filter in a De Morgan lattice L iff L is regular.

Proof. ($\Longrightarrow$) If there exists such a filter T, $x \in T \Leftrightarrow \bar{x} \notin T$ holds for all $x \in L$ (Lemma 2). Hence $x \neq \bar{x}$ for all x, i.e. L is regular.

($\Longleftarrow$) If L is regular then, by (viii) of Lemma 3, $T_1 \cap T_o = \emptyset$. Let $p \in T_1$ (there is such a p because of $x \vee \bar{x} \in T_1$, if $(p \wedge T_1) \cap T_o \neq \emptyset$ there is $q \in T_1$ such that $f_q(p) \in T_o$ (Lemma 5) but, also $f_q(p) \in T_1$ which is a contradiction. Hence, there exists a truth-filter in L. ■

Hence, Belnap and Spencer theorem mentioned at the begining, is proved as a consequence of more general theorem.

Note added in print. The Corollary 1.1 leads to a natural question: Is $a \in I_o$ equivalent to $(a \wedge T_1) \cap T_o \neq \emptyset$? It is obvious that this equivalence could simplify all results and criteria. But the answer is negative. The counter-example is an appropriate complete lattice. The proof can be found in the paper mentioned in the footnote at the begining.

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LOGIC SEMINAR IN BEOGRAD

M. Kapetanović

Seminar in mathematical logic was founded in Mathematical Institute in Beograd more than ten years ago and has been taking place in the Institute ever since. Originally it was a joint seminar in logic, algebra and number theory and the present seminar resembles that one in a sence that the main connections between algebra and logic are systematically explored.

There is a considerable number of regular participants at Seminar meetings (held weekly). They include Institute membres, and teachers from the Faculty of Science and Mathematics as well as from some other faculties and also from Novi Sad. The current heads of Seminar are prof. dr. S.B.Prešić and doc. dr Ž.Mijajlović.

The range of subjects examined in seminar lectures has been wide and almost no important area of logic has been totally neglected, but the one treated most extensively has certainly been model theory with its various applications. It all started with basic facts on predicate calculus of the first order, valid formulas and axiomatization of some mathematical theories. The essence and the central role of the compactness theorem was made clear and after that the content of several chapters from the book "Model Theory" of Chang and Keisler was presented in all details.

Another interesting subject that has been payed attention to is nonclassical logic including intuitionistic, relevant and modal calculi and taking care of both semantic and proof-theoretic aspect. The importance of algebraic theories that arise naturally in this context, such as Boolean and Heyting algebra and Lindenbaum algebras in general, has been pointed out all the time and their properties considered. The notion of the so called reproductive solution of an equation was used by S.B.Prešić to develop a method of solving some Boolean (and even more general) equations. Recently the idea has been put into work of treating propositional calculi as special first order theories with equality.

Something must be said about the leading role of prof. S.B.Prešić in Seminar. It is mainly due to his enthusiasm, guidance and pedagogical work that Seminar has been advancing steadily.

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FUNCTIONAL EQUATION OF GENERALIZED ASSOCIATIVITY

A. Krapež

The functional equation of associativity, either in its most general form or in some special case, has been studied by many mathematicians: Suškevič; Aczél, Belousov, Hosszú; Belousov; Schauffler; Devidé; Prešić; Milić; and others.

The most striking of all results about associativity equation is probably the Four quasigroups theorem (see [1]).

Here we give the general solution of generalized associativity without any assumptions about functions involved.

<u>THEOREM 1.</u> The general solution (on a nonempty set S) of the generalized associativity equation

$$A(x, B(y,z)) = C(D(x,y), z) \tag{1}$$

is given by:

$$A(x,y) = (fy)(x)$$

$$B(x,y) = P(x,y)$$

$$C(x,y) = (gx)(y)$$

$$D(x,y) = Q(x,y)$$

where:

(a) P and Q are arbitrary groupoids on S,

(b) T is an arbitrary 3-groupoid on S, such that

$$\ker P \,\square\, \ker Q \subset \ker T,$$

(c) $f: S \to \mathcal{T}_S$ and $g: S \to \mathcal{T}_S$ are arbitrary functions from S to the set $\mathcal{T}_S$ of all transformations on S, such that

$$fP(x,y) = \rho_{xy}, \qquad gQ(x,y) = \lambda_{xy},$$

ρ and λ being defined by:

$$\lambda_{xy} z = \rho_{yz} x = T(x,y,z).$$

If α and γ are equivalence relations on S^2, then:

$$(x,y,z)\ \alpha_1\ (u,v,w) \iff x = u \wedge (y,z)\ \alpha\ (v,w)$$

$$(x,y,z)\ \gamma_3\ (u,v,w) \iff (x,y)\ \gamma\ (u,v) \wedge z = w$$

and

$$\alpha \,\square\, \gamma = \alpha_1 \vee \gamma_3.$$

The following theorem, giving the associativity criterion for binary operations, is an easy consequence of Th1.

THEOREM 2. Let A be a binary operation on S and $T(x,y,z) = A(x, A(y,z))$. Then, A is associative iff $g_o = \{(A(x,y), \lambda_{xy}) \mid x,y \in S\}$ is a function from $A(S,S)$ to $\mathcal{T}_S$ and $(g_o x)(y) = A(x,y)$ for all $x \in A(S,S)$ and $y \in S$.

Also, from Th1 we can easily prove the Four quasigroups theorem if we assume that A,B,C,D in (1) are quasigroups.

In [2], Schauffler proved the following theorem:

Theorem 3. For any two quasigroups A and B on S there are quasigroups C and D (on S) such that (1) holds iff $|S| \le 3$.

The following theorem, analogous to Th. 3, is also a consequence of Th. 1.

Theorem 4. For any two groupoids A and B on S there are groupoids C and D (on S) such that (1) holds iff S is infinite or $|S| = 1$.

The proofs and other examples are given in [3].

R E F E R E N C E S

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ON SYMMETRIC WORDS IN NILPOTENT GROUPS

Sava A. Krstić

Let G be a group and $F_G(x_1,...,x_r)$ the group freely generated by $x_1,...,x_r$ in the smallest variety of groups containing G. Let A be the group of automorphisms of $F_G(x_1,...,x_r)$ induced by the mappings

$$x_1 \to x_{\sigma i}, \qquad 1 \le i \le r,$$

where σ runs over the group of permutations of the set $\{1,...,r\}$. Clearly, the set

$$S_G{}^r = \{w \in F_G(x_1,...,x_r) \,|\, w = \alpha w \quad \text{for every} \quad \alpha \in A\}$$

is a group, and we naturally call it the group of symmetric words (or operations) in G in r variables $x_1,...,x_r$. The mapping $\partial^r_{r-1} : S_G{}^r \to S_G{}^{r-1}$ defined by

$$\partial^r_{r-1} \; (w(x_1,...,x_r)) = w(x_1,...,x_{r-1},1)$$

is a homomorphism.

All of the notations we have introduced above one can find in Płonka's articles [1-3]. Among other things, he (in [2] and [3]) showed that ∂^r_{r-1} is in fact an isomorphism for the case of G free nilpotent or nilpotent of class ≤ 3 (in both cases with an indispensable assumption that r is greater than the nilpotency class of G). The following theorem, which is a generalization of these results, was stated in [3] as a problem.

Theorem. Let G be a nilpotent group of class n. For every $r > n$ the mapping ∂^r_{r-1} is an isomorphism.

The complete proof of this theorem that we presented at the Meeting is to apear in Publ. Inst. Math.

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ON THE ACTIVITY OF THE SEMINAR FOR SEMIGROUPS

Dragan Blagojević

It has been done very little in the semigroup theory in Belgrade till lately. Dragica Krgović was the only one who has published in that field. Two recent events started the work on semigroups: the first was appearance (defining) of anti-inverse semigroups, and the second was M.Petrich's arrival in Belgrade. S.Milić and his group have worked on anti-inverse semigroups. They have already published some papers on this subject, and they will probably give us detail information about it, since they are present here. I will inform you about the results of Petrich's arrival in Belgrade.

Mario Petrich, one of the world experts for semigroups, was staying in Belgrade during a school-year 1978/79. In Mathematical institute he gave lectures on structure of regular semigroups in two-semester course based on his script "Structure of regular semigroups", Montpelier, 1977. He also held a seminar for amalgamation of semigroups. There were five regular participants to both the course and the seminar: B.Alimpić, D.Krgović, V.Šimić, A.Krapež i D.Blagojević. Petrich also had several consultations with participants of the seminar. All of us agreed that Petrich's visit to Belgrade was very useful because this excellent expert was always ready to direct attendants in their work, to make comments and to help, whenever he was asked to.

After his departure we decided to continue with our work in the semigroups. B.Alimpić became a new chief of the seminar. Our method of work is following: we choose an interesting subject, somebody presents principal known ideas about it, and then we go on studying and reading papers, constantly trying to find out something new, if possible. We also try to keep connection with Petrich.

Since the seminar has begun only recently and because of the fact that we are all the beginners in semigroups, it is self-understanding that there has not been any important result so far. The present participants of the seminar have either exposed the results of the studies they have been engaged in till now, or they will do it soon. Now absent Vida Šimić worked on the equations in the free semigroups, especially on the equations of the form $w = \overline{w}$, where $\overline{w}$ is the mirror image of the word w. I have worked in formal language theory up to now, focusing my attention on normalization of grammars.

Unfortunately, we must admit that there has not been enough cooperation between us and the group I mentioned at the begining. We can not set the blame on one side only, and we shall try to improve this cooperation.

We also cordially invite algebraists from other mathematical centers in Yugoslavia to cooperate with us in the field of semigroups.

Thank you.

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ON BI-IDEALS IN SEMIGROUPS

Dragica N. Krgović

If Ω is the translational hull of a semigroup S, then the set Π of all inner bitranslations is an ideal of Ω. Clearly, $\Pi \subset \Gamma \times \Delta$, where Γ and Δ are the sets of all inner left and of all inner right translations of S. In this paper we prove that $\Omega \cap (\Gamma \times \Delta)$ is an ideal of Ω. It will be denoted by $\overline{\Pi}$. Introducing the concept of the bi-idealizer of a subsemigroup in S, we determine the bi-idealizer of $\overline{\Pi}$ in $\Lambda \times P$. If S is a globally idempotent or reductive semigroup, then Ω is the bi-idealizer of $\overline{\Pi}$ in $\Lambda \times P$.

The reader is refered to [1] and [2] for all concepts not defined in the paper.

Let A be a subsemigroup of a semigroup S. The greatest subsemigroup of S having A as an ideal [bi-ideal] is the idealizer [bi-idealizer] of A in S, to be denoted by $i_S(A)$ [bi-$i_S(A)$].

Let $\Omega_\ell = \{(\lambda,\rho) \in \Lambda \times P \mid xy(\lambda z) = x(y\rho)z\}$ and $\Omega_r = \{(\lambda,\rho) \in \Lambda \times P \mid x(\lambda y)z = (x\rho)yz\}$ for any $x,y,z \in S$ and $\Omega_o = \Omega_\ell \cap \Omega_r$. It is easy to verify that Ω_ℓ, Ω_r and Ω_o are semigroups.

LEMMA 1. Let I be an ideal of Ω_o such that $\Pi \subset I \subset \Omega$. Then $i_{\Lambda\times P}(I) = \Omega_o$.

Proof. Let $(\lambda,\rho)\in i_{\Lambda\times P}(I)$. Since $\Pi\subset I$ we have $(\lambda,\rho)(\lambda_s,\rho_s)\in I$ and $(\lambda_s,\rho_s)(\lambda,\rho)\in I$ for every $s\in S$. Let $x,y,z\in S$. Then $(\lambda,\rho)(\lambda_y,\rho_y) = (\lambda\lambda_y,\rho\rho_y)\in I\subset\Omega$. We have $x(\lambda\lambda_y z) = (x\rho\rho_y)z$, i.e. $x(\lambda y)z = (x\rho)yz$, for every $x,y,z\in S$. Therefore, $(\lambda,\rho)\in\Omega_r$. Also, $(\lambda_y,\rho_y)(\lambda,\rho) = (\lambda_y\lambda,\rho_y\rho)\in I\subset\Omega$, and we have $x(\lambda_y\lambda z) = (x\rho_y\rho)z$, i.e. $xy(\lambda z) = x(y\rho)z$, for every $x,y,z\in S$. Therefore, $(\lambda,\rho)\in\Omega_\ell$. Thus $(\lambda,\rho)\in\Omega_o$, i.e. $i_{\Lambda\times P}(I) = \Omega_o$.

If S is weakly reductive semigroup, then $\Omega_o = \Omega$. Let S be a globally idempotent semigroup and $(\lambda,\rho)\in\Omega_\ell$, i.e. $xy(\lambda z) = x(y\rho)z$ for every $x,y,z\in S$. For $x,y\in S$, we have

$$x(\lambda y) = (uv)(\lambda y) = uv(\lambda y) = u(v\rho)y = (uv)\rho y = (x\rho)y.$$

Thus $(\lambda,\rho)\in\Omega$. Therefore, $\Omega_\ell\subset\Omega$, which implies $\Omega_\ell = \Omega$. Thus, if S is a globally idempotent semigroup, then $\Omega_\ell = \Omega_r = \Omega$. According to Lemma 1, we have

LEMMA 2. Let S be a weakly reductive or a globally idempotent semigroup. If I is an ideal of Ω such that $\Pi\subset I$, then Ω is the idealizer of I in $\Lambda\times P$.

PROPOSITION. 1* If S is a weakly reductive or a globally idempotent semigroup, then

$$i_{\lambda\times P}(\Pi) = i_{\lambda\times P}(\Omega) = \Omega.$$

THEOREM 1. In any semigroup S the following hold:

i) $\Pi\subset\bar{\Pi}$.

ii) $\bar{\Pi}$ is an ideal of Ω.

* See [2], V.1.11. Proposition.

iii) If S is left or right reductive semigroup, then $\Pi = \bar{\Pi}$.

Proof. i) It is easy to verify that

$$\Pi = \{(\lambda_s,\rho_t) \mid (\forall x,y\in S)(xsy=xty)\}$$

Thus $(\lambda_s,\rho_s)\in\bar{\Pi}$ for every $s\in S$.

ii) Let $(\lambda_s,\rho_t)\in\bar{\Pi}$ and $(\lambda,\rho)\in\Omega$. Then $(\lambda_s,\rho_t)(\lambda,\rho) = (\lambda_s\lambda,\rho_t\rho) = (\lambda_{s\rho},\rho_{t\rho})$ and $x(s\rho)y = xs(\lambda y) = xt(\lambda y) = x(t\rho)y$ for every $x,y\in S$. Therefore, $(\lambda_{s\rho},\rho_{t\rho})\in\bar{\Pi}$. Also, $(\lambda,\rho)(\lambda_s,\rho_t) = (\lambda_{\lambda s},\rho_{\lambda t})\in\bar{\Pi}$ is proved analogously. Thus, $\bar{\Pi}$ is an ideal of Ω.

iii) Let S be a left reductive semigroup and $(\lambda_s,\rho_t)\in\bar{\Pi}$. By left reductivity, we have

$$(\forall x,y\in S)(xsy=xty) \Rightarrow (\forall y\in S)(sy=ty) \Rightarrow \lambda_s = \lambda_t.$$

Thus $(\lambda_s,\rho_t) = (\lambda_t,\rho_t)\in\Pi$, so that $\bar{\Pi}\subset\Pi$. By i) we have $\bar{\Pi} = \Pi$. If S is right reductive, then $(\lambda_s,\rho_t) = (\lambda_s,\rho_s)\in\Pi$.

According to Lemma 2 and Theorem 1, we have

COROLLARY 1. If S is a weakly reductive or a globally idempotent semigroup, then $i_{\Gamma\times\Delta}(\Pi) = \bar{\Pi}$ and $i_{\Lambda\times P}(\bar{\Pi}) = \Omega$.

It is easy to verify that the next statement holds:

PROPOSITION 2. The following conditions on a semigroup S are equivalent:

i) S is reductive.

ii) $(\lambda_a,\rho_b)\in\Pi \Rightarrow a = b$.

iii) $(\lambda_a,\rho_b)\in\bar{\Pi} \Rightarrow a = b$.

There exist semigroups for which $\Pi\neq\bar{\Pi}$. For example, consider the next semigroups:

	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	a	a	a
d	a	a	c	d

$\overline{\Pi} = \Pi \cup \{(\lambda_b, \rho_c)\}$
$\overline{\Pi} \neq \Pi$

— S is a weakly reductive semigroup.

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

$\overline{\Pi} = \Pi \cup \{(\lambda_a, \rho_c), (\lambda_c, \rho_a)\}$
$\overline{\Pi} \neq \Pi$

— S is not a weakly reductive semigroup.

In the next examples we have $\Pi = \overline{\Pi}$.

	a	b	c	d
a	a	a	a	a
b	a	a	b	b
c	a	a	c	d
d	a	a	c	d

— S is a weakly reductive but it is neither left nor reductive semigroup.

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

— S is not a weakly reductive semigroup.

Let $\overline{\Omega} = \{(\lambda,\rho) \in \Lambda \times P \mid xy(\lambda z)u = x(y\rho)zu\}$. $\overline{\Omega}$ is a subsemigroup of $\Lambda \times P$.

PROPOSITION 3. In any semigroup S we have

i) Π is a bi-ideal of Ω_ℓ, Ω_r and Ω_o.

ii) $\overline{\Pi}$ is a bi-ideal of Ω_ℓ, Ω_r, Ω_o and $\overline{\Omega}$.

Proof. i) Let $(\lambda,\rho) \in \Omega_\ell$ and $(\lambda_s,\rho_s),(\lambda_t,\rho_t) \in \Pi$. Then $(\lambda,\rho) \in \Omega_\ell \implies (\forall x,y,z \in S)\,(xy(\lambda z) = x(y\rho)z) \implies$
$\implies (\forall y,z \in S)\,(\rho_{y(\lambda z)} = \rho_{(y\rho)z})$.

Thus $(\lambda_s,\rho_s)(\lambda,\rho)(\lambda_t,\rho_t) = (\lambda_{s(\lambda t)}, \rho_{(s\rho)t}) =$
$= (\lambda_{s(\lambda t)}, \rho_{s(\lambda t)}) \in \Pi$. Therefore, Π is a bi-ideal of Ω_ℓ.

Analogously, Π is a bi-ideal of Ω_r so that Π is a bi-ideal of Ω_o.

ii) Since $\Omega_o \subset \Omega \subset \overline{\Omega}$, we prove that $\overline{\Pi}$ is a bi-ideal of $\overline{\Omega}$. Let $(\lambda,\rho) \in \overline{\Omega}$, i.e. $xy(\lambda z)u = x(y\rho)zu$ for every $x,y,z,u \in S$. Let $(\lambda_s,\rho_t),(\lambda_u,\rho_v) \in \overline{\Pi}$. Then $(\lambda_s,\rho_t)(\lambda,\rho)(\lambda_u,\rho_v) = (\lambda_{s(\lambda u)}, \rho_{(t\rho)v})$ and $xs(\lambda u)y = xt(\lambda u)y = x(t\rho)uy = x(t\rho)vy$ for every $x,y \in S$. Therefore, $\overline{\Pi}$ is a bi-ideal of $\overline{\Omega}$.

LEMMA 3. If B is a bi-ideal of $\overline{\Omega}$ such that $\Pi \subset B \subset \Omega$, then

$$\text{bi-i}(B) = \overline{\Omega}.$$

Proof. Let $(\lambda,\rho) \in \text{bi-i}_{\Lambda\times P}(B)$. Then $(\lambda_s,\rho_s)(\lambda,\rho)\cdot(\lambda_t,\rho_t) \in B$ for every $s,t \in S$. Therefore, $(\lambda_{s(\lambda t)}, \rho_{(s\rho)t}) \in B \subset \Omega$. Thus, $xs(\lambda t)y = x(s\rho)ty$ for every $x,y,s,t \in S$, i.e. $(\lambda,\rho) \in \overline{\Omega}$. Since B is a bi-ideal of $\overline{\Omega}$ we have $\overline{\Omega}$ is a bi-idealizer of B in $\Lambda \times P$.

According to Proposition 3 and Lemma 3, we have

PROPOSITION 4. If S is a semigroup, then $\overline{\Omega}$ is the bi-idealizer of $\overline{\Pi}$ in $\Lambda \times P$.

If S is a reductive or a globally idempotent semigroup, then $\overline{\Omega} = \Omega$.

COROLLARY 2. Let S be a semigroup. Then

i) If S is reductive, then $\text{bi-i}_{\Lambda\times P}(\Pi) = \Omega$.

ii) If S is a globally idempotent semigroup, then $\text{bi-i}_{\Lambda\times P}(\overline{\Pi}) = \Omega$.

Let $\mathcal{F}(X)$ be the full transformation semigroup on a set X (the semigroup operation is composition of mappings: if $\alpha,\beta \in \mathcal{F}(X)$, then $\alpha\circ\beta$ is defined by

$x(\alpha o \beta) = (x\alpha)\beta$ $(x \in X))$. Let $\mathcal{F}^*(X)$ be the dual semigroup of $\mathcal{F}(X)$ and

$$\mathcal{L}(S) = \{(\lambda,\rho) \in \mathcal{F}^*(S) \times \mathcal{F}(S) \mid x(\lambda y) = (x\rho)y\}.$$

THEOREM 2. In any semigroup S the following hold:

i) $\mathcal{L}(S)$ is a semigroup.

ii) $\lambda_s\lambda = \lambda_{s\rho}$, $\rho\rho_s = \rho_{\lambda s}$, for every $s \in S$ and $(\lambda,\rho) \in \mathcal{L}(S)$.

iii) Π is a bi-ideal of $\mathcal{L}(S)$.

iv) $\overline{\Pi}$ is a bi-ideal of $\mathcal{L}(S)$.

<u>Proof.</u> i) By a straightforward verification.

ii) Let $s,x \in S$. Then $(\lambda_s\lambda)x = \lambda_s(\lambda x) = s(\lambda x) = (s\rho)x = \lambda_{s\rho}x$, $x(\rho\rho_s) = (x\rho)\rho_s = (x\rho)s = x(\lambda s) = x\rho_{\lambda s}$.

iii) Let (λ_s,ρ_s), $(\lambda_t,\rho_t) \in \Pi$ and $(\lambda,\rho) \in \mathcal{L}(S)$. Then

$$(\lambda_s,\rho_s)(\lambda,\rho)(\lambda_t,\rho_t) = (\lambda_s\lambda\lambda_t,\rho_s\rho\rho_t).$$

According to ii), we have

$$\lambda_s\lambda\lambda_t = \lambda_{s\rho}\lambda_t = \lambda_{(s\rho)t},$$

$$\rho_s\rho\rho_t = \rho_s\rho_{\lambda t} = \rho_{s(\lambda t)}.$$

Since $(s\rho)t = s(\lambda t)$, we have $(\lambda_{(s\rho)t}, \rho_{s(\lambda t)}) \in \Pi$. Thus Π is a bi-ideal of $\mathcal{L}(S)$.

iv) Let (λ_s,ρ_t), $(\lambda_u,\rho_v) \in \overline{\Pi}$ and $(\lambda,\rho) \in \mathcal{L}(S)$, i.e. $xsy = xty$, $xuy = xvy$ and $x(\lambda y) = (x\rho)y$ for every $x,y \in S$. According to ii), $(\lambda_s,\rho_t)(\lambda,\rho)(\lambda_u,\rho_v) = (\lambda_s\lambda\lambda_u,\rho_t\rho\rho_v) = (\lambda_{(s\rho)u},\rho_{t(\lambda v)})$. Then $x(s\rho)uy = x(s\rho)vy = xs(\lambda v)y = xt(\lambda v)y$ for every $x,y \in S$. Therefore, $(\lambda_{(s\rho)u}, \rho_{t(\lambda v)}) \in \overline{\Pi}$.

LEMMA 4. Let S be a globally idempotent semigroup and let B be a bi-ideal of $\mathcal{L}(S)$ such that $\Pi \subset B \subset \Omega$. Then

i) $\text{bi-i}_{\mathcal{F}^* \times \mathcal{F}}(B) = \mathcal{L}(S)$

ii) $\text{bi-i}_{\Lambda \times P}(B) = \Omega$.

<u>Proof.</u> Let (λ,ρ) $\text{bi-i}_{\mathcal{F}^* \times \mathcal{F}}(B)$. Then $(\lambda_s,\rho_s)\cdot(\lambda,\rho)(\lambda_t,\rho_t) \in B$ for every $s,t \in S$. Then $(\lambda_s\lambda\lambda_t, \rho_s\rho\rho_t) \in B \subset \Omega$. Thus $x(\lambda_s\lambda\lambda_t y) = (x\rho_s\rho\rho_t)y$ and $xs\lambda(ty) = (xs)\rho ty$ for every $x,y,s,t \in S$. Since S is a globally idempotent semigroup we have $(\lambda,\rho) \in \mathcal{L}(S)$, i.e. i) holds.

ii) Evident by i).

According to Theorem 2 and Lemma 4, we have

COROLLARY 3. Let S be a globally idempotent semigroup. Then

i) $\text{bi-i}_{\mathcal{F}^* \times \mathcal{F}}(\Pi) = \text{bi-i}_{\mathcal{F}^* \times \mathcal{F}}(\overline{\Pi}) = \mathcal{L}(S)$.

ii) $\text{bi-i}_{\Lambda \times P}(\Pi) = \text{bi-i}_{\Lambda \times P}(\overline{\Pi}) = \Omega$.

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INTERPOLATION FORMULAS OVER FINITE SETS

Koriolan Gilezan

Let $E_1,\dots,E_n$ be n finite nonempty sets (not necessarily pairwise distinct) and $p = |E_1 \times \dots \times E_n|$. Further, let E be a set containing two distinguished elements ω and ε, with $\omega \neq \varepsilon$, and endowed with a unary operation $*: E \to E$, a partially defined p-ary operation $\circ: E^p \xrightarrow{\circ} E$ and a partially defined (n+1)-ary operation $\bullet: E^{n+1} \xrightarrow{\bullet} E$. Whenever the values $\circ(y_1,\dots,y_p)$ and $\bullet(z_o,z_1,\dots,z_n)$ are defined, they will be denoted by $y_1 \circ y_2 \circ \dots \circ y_p$ and $z_o \bullet z_1 \bullet \dots \bullet z_n$, respectively. Finally suppose the given operations fulfil:

$$(y_1 \bullet \beta_{11} \bullet \dots \bullet \beta_{1n}) \circ \dots \circ (y_{i-1} \bullet \beta_{i-1,1} \bullet \dots \bullet \beta_{i-1,n}) \circ$$
$$\circ (y_i^* \bullet \varepsilon \bullet \dots \bullet \varepsilon) \circ (y_{i+1} \bullet \beta_{i+1,1} \bullet \dots \bullet \beta_{i+1,n}) \circ \dots \tag{1}$$
$$\dots \circ (y_p \bullet \beta_{p1} \bullet \dots \bullet \beta_{pn}) = y_i$$

for any $y_1,\dots,y_p \in E$, $\beta_{hk} \in \{\omega,\varepsilon\}$ $(h = 1,\dots,i-1,i+1,\dots,p;\ k = 1,\dots,n)$ such that $\beta_{hk(h)} = \omega$ for all $h = 1,\dots,i-1,i+1,\dots,p$ and some $k(h) \in \{1,\dots,n\}$, and for all $i = 1,\dots,p$. The exact meaning of this axiom is that the elements $(y_h \bullet \beta_{h1} \bullet \dots \bullet \beta_{hn})$ $(h = 1,\dots,i-1, i+1,\dots,p)$, $(y_i^* \bullet \varepsilon \bullet \dots \bullet \varepsilon)$, as well as the left side of (1), exist and the equality holds.

THEOREM. Under the above assumptions every function

$$f : E_1 \times \ldots \times E_n \rightarrow E$$

fulfils the identity

$$f(x_1,\ldots,x_n) =$$
$$= \underset{\alpha_1 \in E_1,\ldots,\alpha_n \in E_n}{\mathrm{O}} \left[f(\alpha_1,\ldots,\alpha_n))*\cdot x_1^{\alpha_1} \bullet \ldots \bullet x_n^{\alpha_n} \right].$$

Here, the right side stands for the result of the operation O applied to the p elements of the form indicated within the brackets, while for every $x,\alpha \in E_k$ $(k = 1,\ldots,n)$, x^α is defined by

$$x^\alpha = \begin{cases} \varepsilon & \text{if } x = \alpha, \\ \omega & \text{if } x \neq \alpha. \end{cases}$$

This theorem is proved in:

C.Ghilezan, S.Rudeanu, Interpolation formulas over finits sets, Publ. Inst. Math. Nouvelle serie, tome 25 (39), 1979, pp. 45-49

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PARTIAL QUASIGROUPS

Janez Ušan, Zoran Stojaković

In this paper some investigations on partial quasigroups and some related structures are described. The paper is based on the work of a group of mathematicians from Novi Sad whose papers are listed at the end of the paper. The emphasis in the paper is not on listing details and theorems (which the reader may find in the literature), but on exposing general ideas and motives for an investigation of certain structures and on formulating some problems for further work.

1. Partial quasigroups are extensively investigated, but the main direction in these investigations is the problem of embedding. The question when a partial quasigroup can be embedded in a quasigroup, the connection between the order of a partial quasigroup and its domain and the order of the quasigroup in wich the partial quasigroup is embedded, represents the main problem in the investigation of partial quasigroups.

Here we give another direction in the investigation of partial quasigroups. The quasigroups are closely related to many other structures - for example, to geometric nets, affine and projective planes, codes, graphs, various experimental and statistical designs etc. Some of these structures are completely characterized by systems of quasigroups of certain type. However, most of the mentioned structures can be naturally defined and they exist (possibly a little weakened) also in the cases when there are no quasigroups which correspond to them. It can be shown that in this case these structures are characterized by systems of partial quasigroups, and the properties of these structures can be determined from the corresponding properties of partial quasigroups. This kind of investigation is the subject of the present paper and the papers listed at the end.

2. Because of the limited space we shall not give here the definitions of well known basic notions, such as quasigroup, isotopy, orthogonality etc., which can be found in [14], [15], [16] and [17].

Probably the best known is the connection between quasigroups and geometric nets. We give the definition of a k-net:

Let T be a nonempty set and let $L_1,\dots,L_k$, $k \geq 3$, be nonempty mutually disjoint families of subsets of the set T. The elements of T we call points and the elements of the sets $L_1,\dots,L_k$ we call lines (or blocks). Then $(T,L_1,\dots,L_k)$ is said to be a k-net iff the following holds

a) every two lines from different classes L_i, L_j $(i,j \in \{1,\dots,k\} = N_k)$ have one and only one common point,

b) every point from T belongs to one and only one line from every class L_i $(i \in N_k)$.

If T is a finite set, then all lines have the same number n of points, each of the classes $L_i (i \in N_k)$ has exactly n elements and T has n^2 points. n is called the order of the net.

A special case of the geometric net is an affine plane: a n+1-net of order n is an affine plane. From an affine plane by a simple procedure a projective plane is obtained ([15], [17], [18]).

Nets and quasigroups are very closely related, namely, to every quasigroup (of order q) there corresponds (by a procedure described in [15], [17], [18]) a 3-net (of order q). Conversely, to every 3-net different coordinate quasigroups can be associated, but all these quasigroups are isostrophic, so, to every 3-net there corresponds a class of isostrophic quasigroups.

Algebraic properties of quasigroups completely determine geometric properties of the corresponding nets, so these geometric properties can be successfully investigated using the theory of quasigroups. That is the reason why some authors call geometric nets algebraic nets ([17]).

Orthogonal systems of quasigroups have the fundamental role in the investigation of geometric nets (and also in the investigations of many other mathematical structures). Namely, every k-net of order n defines and is defined by a corresponding system of k-2 orthogonal quasigroups of order n. From there it follows that affine and projective planes can also be completely characterized by orthogonal systems of quasigroups (by a well known procedure of M. Hall).

Besides the described connection between quasigroups and nets, there exists very close connection between quasigroups and one kind of codes.

If Q is a nonempty set, then every subset K of the set Q^k is called a k-code. The elements of K we call words, the set Q is called an alphabet and its elements are called letters. The words $(a_1,\ldots,a_k)$, $(b_1,\ldots,b_k)$ from K are said to be on a distance (Hamming distance) d iff they have exactly d different components. If d is the minimum of distances of different elements from K, then we say that the code K is of the code distance d.

It can be shown that to each quasigroup of order q there corresponds a 3-code with q^2 words of code distance 2. As it is the case with geometric structures, orthogonal systems of quasigroups have very important role in the study of certain classes of codes. One way to establich connection between orthogonal systems of quasigroups and a class of codes is given by the following proposition.

To each system of k orthogonal quasigroups of order q there corresponds a k+2-code with q^2 words over an alphabet of q letters of code distance k+1 and vice versa.

In a similar way the connections between quasigroups and other structures can be established. For example, closely related to quasigroups are block designs and various statistical designs. In the graph theory quasigroups are also used, the complete oriented graphs are characterized by horizontally complete quasigroups.

3. As we said before, all these structures (possibly a little weakened) can be defined also when there do not exist quasigroups which characterize them. From an orthogonal system of quasigroups a code with q^2 words is obtained, but codes which do not have q^2 words are extensively used. There exist much more "noncomplete" graphs then complete. The question is why geometric nets (and also affine and projective planes) should have the same number of lines in every family of lines?

In all these casses (and also in some other) we naturally obtain partial quasigroups by which these structures can be characterized.

First we shall define k-seminets [1], which represent a generalization of k-nets.

If in the definition of k-net the condition (a) we replace by the condition:

(a´) every two lines from different classes L_i, L_j $(i,j\in N_k)$ have at most one common point,

then the structure $(T,L_1,\ldots,L_k)$ which satisfies the conditions (a´) and (b) we call a k-seminet.

The number $m = \max\{\mathrm{card}L_i \mid i\in N_k\}$ is called the L-order, and the number $n = \max\{\mathrm{card}\,\ell \mid \ell\in L_1\cup\ldots\cup L_k\}$ is called the T-order of the k-seminet. It is easy to prove that in every k-seminet $n \le m$. L-order and T-order generalize the concept of the order of the k-net.

3-seminets are special casses of halfnets of V. Havel [2].

In order to get an algebraic characterization of k-seminets we need the following definitions ([1], [5]).

Let Q be a nonempty set and $D \subseteq Q \times Q$, $D \neq \emptyset$. If A is a mapping of D into Q, then (Q,A) is said to be a partial groupoid.

A partial quasigroup is a partial groupoid (Q,A) such that if the equations $A(x,b) = c$ and $A(a,y) = c$ have solutions for x and y in Q, then these solutions are unique.

Let (Q,A) and (Q,B) be partial groupoids of the same domain $D = \mathcal{D}A = \mathcal{D}B$, $D \subseteq Q \times Q$. A and B are said to be orthogonal iff for every $a,b \in Q$ for which the system of equations

$$A(x,y) = a, \quad B(x,y) = b,$$

has a solution, this solution is unique.

If (Q,A) is a partial groupoid such that $\operatorname{card}\mathcal{D}A = p$, then if $p \leq q$ it is possible that (Q,A) be orthogonal to itself. If $p > q$, two orthogonal partial groupoids are always different.

If we introduce $O_{AB}(x,y) \overset{\text{def}}{=\!=} (A(x,y), B(x,y))$, then A and B can be said to be orthogonal iff O_{AB} is a bijection of the set D on the set $\mathcal{R}O_{AB}$ (by $\mathcal{R}O_{AB}$ we denote the range of O_{AB}).

The orthogonal partial operations A and B are said to be regularly orthogonal iff for every $(i,j) \in \mathcal{R}O_{AB}$ there exists $j' \in Q$, $j' \neq j$, such that $(i,j') \in \mathcal{R}O_{AB}$ or there exists $i' \in Q$, $i' \neq i$, such that $(i',j) \in \mathcal{R}O_{AB}$.

A partial quasigroup (Q,A) is regular iff the following conditions are satisfied:

$$(\forall(i,j))\Big((i,j) \in D \implies [(\exists j')(j \neq j' \wedge (i,j') \in D)] \vee \vee[(\exists i')(i = i' \wedge (i',j) \in D)]\Big),$$

$$(\forall(i,j)\Big[A(i,j) = t \implies ([\{(i,j)\} = (\{i\} \times Q) \cap D] \vee \vee[\{(i,j)\} = (Q \times \{j\}) \cap D] \implies \implies (\exists(i',j'))((i,j) \neq (i',j') \wedge A(i',j') = t))\Big].$$

The set of different partial operations of the same domain is said to be an orthogonal system of partial operations (ISPO) if each pair of the operations from this set is orthogonal. If each pair of the operations is regularly orthogonal we call such a system a regularly orthogonal system of partial operations (ROSPO).

Some properties of partial quasigroups are considered in [5]; here we give only some theorems on characterization of k-seminets.

<u>Theorem 1.</u> ([1]) To each 3-seminet there corresponds a regular partial quasigroup and vice versa.

<u>Theorem 2.</u> ([1]) To each k-seminet, $k > 3$, there corresponds a regularly orthogonal system of $k-2$ regular partial quasigroups and vice versa.

The m-dimensional k-seminets are already being studied.

A class of codes can be also characterized using partial quasigroups. The following theorem generalizes the main result from [6].

<u>Theorem 3.</u> ([5]) To every k-code of p elements,

$k \geq 2$, of code distance $k-1$ over an alphabet of q lettres, $q < p$, there corresponds an orthogonal system of partial quasigroups (OSPQ) of $k-2$ elements defined on a set of q elements, with the domain of p elements, $q < p$, and vice versa.

The preceding theorem can be generalized for n-ary partial quasigroups (see [5]).

A procedure for a construction of systems of latin rectangles (which are a special case of partial quasigroups), codes and k-seminets is given in [3]. The main result which enables that procedure is the following theorem (the proof of which is based on a result of Houston ([4]).

Theorem 4. ([3]) If there exists a permutation $(a_0, a_1, \ldots, a_{q-1})$ of numbers $0, 1, \ldots, q-1$, such that for every $k \in N_m$ all members of the sequence $b_i^k = \sum_{j=i}^{k} (a_{j+i} - a_{j+i-1}) \pmod q$, $i = 0, 1, \ldots, q-(k+1)$, are pairwise noncongruent modulo q, then there exists a system with $m+1$ mutually orthogonal $q \times (q-1)$ latin rectangles.

It can be shown that to every k-code of q^2 words, $k \geq 3$, of code distance $k-1$ over an alphabet of q letters there corresponds a k-net. If for some k and q k-nets does not exist, then the maximal k-code of code distance $k-1$ over an alphabet of q elements has the cardinality smaller than q^2. In this case the maximal code should be searched for among the OSPQ of cardinality q defined on a set of q elements. To the maximal k-code then there corresponds an OSPQ of maximal cardinality of the domain of its elements. Does there a k-seminet correspond to the searched OSPQ (and when it does under what conditions)? This question is equivalent to the following: can every OSPQ defined on a finite set Q (and if it can, under what conditions) be embedded in an ROSPPQ defined on Q.

It is accustomed to call an orthogonal system of quasigroups (OSQ) Σ complete iff for every OSQ Σ' we have $\Sigma \subseteq \Sigma' \Rightarrow \Sigma = \Sigma'$. A similar definition could be given for OSPQ.

In [7], [8] and [9] a different kind of "completness" of OSPQ is considered - a "completnes" which is related to the cardinality of the domain of the operations from OSPQ.

An OSPQ $\Sigma = \{A_1, \ldots, A_k\}$ defined on a set Q is called D-complete iff for every OSPQ $\Sigma' = \{\overline{A}_1, \ldots, \overline{A}_k\}$ on Q we have

$$A_1 \subseteq \overline{A}_1 \wedge \ldots \wedge A_k \subseteq \overline{A}_k \Longrightarrow A_1 = \overline{A}_1 \wedge \ldots \wedge A_k = \overline{A}_k .$$

A partial quasigroup (Q,A) is complete iff for every partial quasigroup $(Q,\overline{A})$ we have $A \subseteq \overline{A} \Longrightarrow A = \overline{A}$.

An important class of codes are complete codes.

A k-code K of code distance d over the alphabet Q is said to be complete iff for every k-code $\overline{K}$ of code distance d over the alphabet Q the following holds: $K \subseteq \overline{K} \Longrightarrow K = \overline{K}$.

Theorem 5. ([7]) Let $\Sigma = \{A_1, \ldots, A_k\}$ be a D-complete OSPQ on a set Q with q elements, where $k \leq q-1$. Then, by the construction from [5], the corresponding $k+2$-code of code distance $k+1$ is complete.

To illustrate the questions which were discussed and which are related to D-complete OSPQ we give some theorems from [7] and [9].

Theorem 6. ([7]) If $\Sigma = \{A_1,\dots,A_k\}$ is a D-complete OSPQ on Q, $k \le q-1$, where q=card Q, then card$\mathscr{D}A_i > q$.

Theorem 7. ([9]) For every even natural number q there exists a D-complete OSPQ $\Sigma = \{A_1,A_2\}$ over Q, where cardQ = q, such that the following holds

$$\text{card}\,\mathscr{D}A_1 = \text{card}\,\mathscr{D}A_2 = q(q-1),$$

where neither of the operations A_1 and A_2 is complete.

The preceding theorem is proved using horizontally complete quasigroups.

Some other properties of D-complete OSPQ are determined in [7].

Another question which arises from the relations between k-seminets and partial quasigroups is the following.

Let (Q,A) be a partial groupoid. By A_x we denote the set of all first coordinates of the pairs from A, by A_y the set of all second coordinates and by A_z the range of A. Usually it is "tacitly" assumed that $A_x \cup A_y \cup A_z = Q$. However, in some cases partial quasigroups which do not satisfy this condition appear.

A partial quasigroup (Q,A) is called compressible ([8]) iff

$$A_x \ne Q \wedge A_y \ne Q \wedge A_z \ne Q.$$

The partial quasigroup from Tab. 1. is compressible and regular. To this partial quasigroup corresponds

the 3-seminet from Fig.1. But to this 3-seminet, according to the construction from [1], there corresponds the partial quasigroup from Tab.2. So we could say that the partial quasigroup from Tab.2. is a "compression" of the partial quasigroup from Tab.1.

	1	2	3	4
1	1	2		3
2	3	1		
3		3		2
4				

Tab.1.

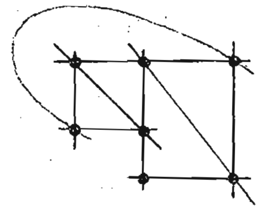

Fig.1.

	a	b	c
a	a	b	c
b	c	a	
c		c	b

Tab.2.

Some questions concerned with compressability of partial quasigroups are discussed in [8] and [7].

At the end we make a remark on embedding of partial [n,m] - quasigroups ([13]).

Let Q be a nonempty set, n,m natural numbers and $f:(x_1,\dots,x_n) \to f(x_1,\dots,x_n)$ a mapping from Q^n into Q^m. (Q,f) is called an [n,m] - quasigroup iff for every injection $\phi: N_n \to N_{n+m}$ there exists a unique sequence $(b_1,\dots,b_{n+m}) \in Q^{n+m}$ such that $b_{\phi(1)}=a_1,\dots,b_{\phi(n)}=a_n$ and

$$f(b_1,\dots,b_n) = (b_{n+1},\dots,b_{n+m}).$$

[n,m] - quasigroups are a convenient generalization of n-ary quasigroups. If we define a partial [n,m] - quasigroup as it is done for partial quasigroups (see [13]), then we have the following:

Theorem 8. ([13]) Every partial [n,m] - quasigroup (Q,f) can be embedded in an [n,m] - quasigroup (Q′,f′).

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BASIS CLASS OF ONE CLASS OF SEMIGROUPS

S. Crvenković

The following definition is given by E.S.Ljapin [3] (VIII, §5):

DEFINITION. Let $\underline{M},\underline{N},\underline{P}$ be three classes of semigroups such that $\underline{M}\subset\underline{N}\subset\underline{P}$. The class $\underline{M}$ is a basis class for the class $\underline{N}$ relative to the class $\underline{P}$ iff the following conditions are satisfied:

a) Every semigroup from $\underline{N}$ can be represented as a union of its subsemigroups which are from the class $\underline{M}$,

b) Every semigroup from $\underline{P}$ which can be represented as a union of its subsemigroups from $\underline{M}$ is in $\underline{N}$,

c) None of the subclasses $\underline{M}'$ of the class $\underline{M}$ satisfies the condition a).

Let S be a semigroup. Denote by $\underline{S}^k$ the class of semigroups such that

$$S\in\underline{S}^k \quad (\forall x\in S)(\exists y\in S)\left(\bigwedge_{i=1}^{k} w_i = v_i\right),$$

where w_i, v_i are elements of the subsemigroup $[\{x,y\}]$ generated by x and y. Take $\underline{M}$ to be the class of semigroups from $\underline{S}^k$ with the following property:

d) In each semigroup U_o of $\underline{M}$ there must exist an element a_o which is not contained in any subsemigroup of the semigroup U_o belonging to $\underline{M}$ and not isomorphic to U_o. Obviously, $\underline{M}$ is the basis class for $\underline{S}^k$ relative to the class of all semigroups.

For $k=3$, $w_1=x^m$, $w_2=yx$, $w_3=x^n$, $v_1=y^m$, $v_2=x^{m+1}y$ and $v_3=x$ we have that $\underline{S}^3=\underline{S}^*_{m,n}$. According to [1] (Theorem 2.1) it follows that $[\{x,y\}]$ is a group. The group $[\{x,y\}]$ is finite and belongs to $\underline{S}^*_{m,n}$ [1] (Theorem 2.3). Take all such groups, that satisfy the condition d), to be the class $\underline{M}$. $\underline{M}$ is the basis class for $\underline{S}^*_{m,n}$ relative to the class of all semigroups.

From $\alpha,\beta \in [\{x,y\}]$ and $\alpha^m=\beta^m$, $\beta\alpha=\alpha^{m+1}\beta$ it follows that $[\{\alpha,\beta\}]$ is a subgroup of $[\{x,y\}]$ [1] (Theorem 2.3). $[\{x,y\}]\in \underline{M}$ iff $[\{\alpha,\beta\}] = [\{x,y\}]$ for all $\alpha,\beta \in [\{x,y\}]$. If $\alpha,\beta \in [\{x,y\}]$ we have that

$$(\exists s,t \in Z)\,\alpha = x^s y^t \quad \text{and} \quad (\exists k,\ell \in Z)\,\beta = x^k y^\ell .$$

$$[\{\alpha,\beta\}] = [\{x,y\}] \Longleftrightarrow (\exists A,B,C,D \in Z)\,(x^s y^t)^A (x^k y^\ell)^B = x \wedge (x^s y^t)^C (x^k y^\ell)^D = y).$$

Cyclic groups $H = \{e,x,x^2,\dots,x^{n-2}\}$ and $K = \{e,y,y^2,\dots,y^{n-2}\}$ are subgroups of the group $[\{x,y\}]$. The intersection

$$P=H\cap K=\{x^{\alpha_1},x^{2\alpha_1},\dots,x^{r\alpha_1}=e\}=\{y^{\beta_1},y^{2\beta_1},\dots,y^{r\beta_1}=e\}$$

determines the number of elements of the group $[\{x,y\}]$.

$$(x^s y^t)^A (x^k y^\ell)^B = x$$

if and only if

$$x^{As+Bk-1}=y^{-(At+B\ell+m\frac{A(A-1)}{2}st+m\frac{B(B-1)}{2}k\ell+ABmkt)}$$

i.e. if and only if there exist q_α, $q_\beta \in Z$ such that

$$As+Bk-1 = q_\alpha \alpha_1$$

$$At+B\ell+m\frac{A(A-1)}{2}st+m\frac{B(B-1)}{2}k\ell+ABmkt = q_\beta\beta_1$$

and

$$x^{q_\alpha\alpha_1}=y^{q_\beta\beta_1}.$$

An algorithm for determination of A and B is given in [2].

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Q_r - SEMIGROUPS

Stojan Bogdanović

A semigroup S is called a Q_r-semigroup if every right ideal of S is a power joind semigroup. Here we give a characterization of weakly commutative Q_r-semigroups.

Theorem. [2]. A semigroup S is a weakly commutative Q_r-semigroup if and only if one of the three possibilities holds:

1° S is a power joind semigroup,

2° S is a group,

3° $S = M \cup G$, where G is a group, the identity e of G is a left identity of S and M is the unique maximal prime ideal of S and power joined.

Commutative Q-semigroups are considered in [5], the results of [5] are extended on quasi-commutative semigroups in [4], and Putcha's Q-semigroups are investigated in [3]. The notion of weakly commutative semigroup was introduces by Petrich in [6].

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ON SOME CLASSES OF SEMIGROUPS

Svetozar Milić

In this paper we give an account of investigations on some subclasses of the class of regular semigroups, i.e. the class of anti-inverse semigroups [1] and classes of (m,n)-anti-inverse semigroups [2],[3], [6].

A semigroup S is <u>regular</u> if

$$(\forall a \in S)(\exists x \in S)(a = axa).$$

The notion of regularity is first introduced by J. von Neumann in [11] for elements of a ring. An element of a ring is regular if it is regular as an element of the multiplicative semigroup of the ring. Regular semigroups under the name of <u>demi-groupes inversifs</u> are considered by G. Thierrin [12]. An important subclass of the class of regular semigroups is the class of inverse semigroups. The notion of an inverse semigroup is first introduced by V. V. Vagner [16] and G. Thierrin [13].

A semigroup S is <u>inverse</u> if

$$(\forall a \in S)(\exists_1 b \in S)(aba = a \wedge bab = b).$$

A semigroup S is anti-inverse if

$$(\forall a \in S)(\exists b \in S)(aba = b \wedge bab = a).$$

The class of anti-inverse semigroups, denoted by $\mathcal{A}$, is a subclass of the class of regular semigroups. Denote that the intersection of the class of anti-inverse semigroups and of the class of inverse semigroups is nonempty. J. S. Sharp [9] the class of anti-inverse semigroups calls anti-regular semigroups.

I. Anti-inverse semigroups

Two elements a,b of a semigroups S are said to be mutually anti-inverse provided that $aba = b$, $bab = a$. A semigroup S is anti-inverse semigroup if each element in S has its own anti-inverse in S. Examples of anti-inverse semigroups are: a left zero semigroup $(xy = x)$, a right zero semigroup $(xy = y)$, a band $(x^2 = x)$, a cyclic group of order 2 and the group of quaternions.

In the following theorem we characterize semigroups in the class $\mathcal{A}$.

Theorem 2.1. [1]

$$S \in \mathcal{A} \iff (\forall x \in S)(\exists y \in S)(x^2=y^2 \wedge yx=x^3y \wedge x^5=x)$$

Theorem 2.2. [1]

$$S \in \mathcal{A} \iff (\forall x \in S)(\exists y \in S)(x^2=y^2 \wedge x^2=(xy)^2 \wedge x^5=x).$$

By Theorem 2.1. we obtain the following

Corollary 2.1. [1]

(i) Every anti-inverse semigroup S is a regular semigroup.

(ii) Each element in S has its own unity.

(iii) Anti-inverse elements from S have the same unity.

(iv) If $x^2 = e_x$ (e_x is its own unity), then the element x is permutable with each of his anti-inverse elements.

(v) If x and y are mutual anti-inverse elements, then $x^2y = yx^2$ and $xy^2 = y^2x$.

(vi) If for $x \in S$ an anti-inverse element is $y \in S$, then so are also: xy, x^2y, x^3y.

(vii) Every anti-inverse semigroup S is an intra-regular semigroup [7].

For details of proofs of Theorem 2.1., Theorem 2.2. and Corollary 2.1. see [1].

We give the following two consequences of the above theorems.

Theorem [1; p.23][9]. Let S be a semigroup. Each element of S has a unique anti-inverse element in S if and only if S is an idempotent semigroup (band).

Proof. Sufficiency. Let S be an idempotent semigroup, then each element x of S is its own anti-inverse. Let us suppose that for x anti-inverse element is some $y \neq x$. Then $x = x^2 = y^2$ (Theorem 2.1.)$=$ $= y$, thus $x=y$, a contradiction.

Necessity. If y is the unique anti-inverse for x, then in virtue of the Corollary 2.1.(vi), xy is also anti-inverse of x and so $y = xy$. Since $x^2 = y^2 = (xy)^2$ (Theorem 2.2.), then multiplying $y = xy$ by y from the right we get $y^2 = xy^2$, i.e. $x^2 = x^3$ which multiplied by x^3 yields $x^5 = x^5x$ and this in view of $x^5 = x$ (Theorem 2.2.) yields the requested idempotency $x = x^2$.

Theorem [1; p. 23][9]. Let S be a semigroup. Any two elements of S are anti-inverses if and only if S is an abelian group in which each element is its own (group) inverse.

Proof. Let S be a semigroup in which any two elements are mutually anti-inverse. Then, by Theorem 2.1. we have

$$(\forall x)(\forall y)(x^2 = y^2,\ yx = x^3y,\ x^5 = x)$$

whence, for x = y we have $x^2 = x^4$, i.e. $x^3 = x$, so

$$(\forall x)(\forall y)(xy = yx).$$

Since $(\forall x \in S)(x^4 = e)$ (e is the unity, Corollary 2.1.(iii)) we have $x^3 = x^{-1}$, i.e. $x = x^{-1}$.

Conversely, as S is an abelian group in which $x = x^{-1}$, we have $(\forall x \in S)(x^2 = e)$ where e is the unity of the group S. Then

$$(\forall x)(\forall y)(x^2 = y^2 = (xy)^2 = e)$$

and by Theorem 2.2. the proof is completed.

Let P be a nonempty subset of a semigroup S. Denote by [P] the subsemigroup of S generated by the set P. Denote by A_a the set of all anti-inverse elements of the element a in the semigroup S, i.e.

$$A_a = \{x \in S \mid axa = x \wedge xax = a\}.$$

Theorem 3.1. [1]. Let S be an anti-inverse semigroup and $a \in S$. Then for each subset $I_a \subset A_a$, $GI_a := [a \cup I_a]$ is a subgroup of S.

By the theorem 3.1. we obtain

(i) If the set I_a has exactly one member and $a \in A_a$, then GI_a is the quaternion group.

(ii) If $a \in I_a$ and I_a has exactly two members, i.e. $I_a = \{a,b\}$, then GI_a is the Klein group or the cyclic group of order 2.

(iii) If $I_a = \emptyset$ and $a^2 \neq e_a$, the group GI_a is the cyclic group of order 4.

The groups in the cases (i), (ii) are anti-inverse, while the group in the case (iii) is not anti-inverse.

By the theorem 3.1. we conclude that every anti-inverse semigroup S is covered by groups, i.e.

$$S = \bigcup_{a \in S} GI_a$$

The following theorem gives a necessary and sufficient condition for a group G to be in $\mathcal{A}$.

Theorem 3.3. [1]. Let G be a group. Then

$$G \in \mathcal{A} \iff (\forall x \in G)(\exists y \in G)([\{x,y\}] \in \mathcal{A}).$$

From the proof of theorem 3.1. we have that [{x,y}] is either the group of quaternions or the Klein group (which is the union of cyclic groups of order 2) or a cyclic group of order 2, or the trivial group.

Note that the theorem 3.3. is valid if G is a semigroup [5].

The following definition is given by E. S. Lapin (Chapter VIII §5 [8]).

Definition (basic class). Let M,N,P be three classes of semigroups such that $M \subset N \subset P$. The class M is basic class for the class N, relatively to the class P, if the following conditions hold:

a) Every semigroup from N can be represented as the union of its subsemigroups which are from the class M.

b) Every semigroup from P which can be represented as the union of its subsemigroups from M is in N.

c) Any subclass M of the class M does not satisfy the condition a).

For the class of semigroups having the basic class in the sense of previous definition, we shall say to have basic class in the Lapin's sense.

Denote by $\mathcal{B}$ the class consisting of the trivial group, cyclic group of order 2 and of quaternion group.

If $\underline{M} = \mathcal{B}$, $\underline{N} = \mathcal{A}$, $\underline{P} = \mathcal{S}$ and $\mathcal{S}$ is the class of all semigroups, then we have:

Theorem 3.3. [5] The class $\mathcal{B}$ is a basic class in the sense of Lapin for the class $\mathcal{A}$ relatively to $\mathcal{S}$.

Different characterizations of anti-inverse semigroups by the aim of ideals can be found in [4], [5].

II. (m,n)-anti-inverse semigroups

In the paper [2] are cosidered semigroups satisfying

(1) $$(\forall x)(\exists y)(x^m = y^m \wedge x^m = (xy)^m \wedge x^n = x)$$

where $m,n \in \mathbb{N}$. We denote by $\mathcal{S}_{m,n}$ the class of semigroups for which (1) holds. It is clear that class of anti-inverse semigroups is the same as $\mathcal{S}_{2,5}$, i.e. $\mathcal{A} = \mathcal{S}_{2,5}$.

The following question arrises: for which m,n a semigroup S satisfying the condition (1) is an anti-inverse semigroup?

The answer of this question is given in [2], where an algorithm is given, by which, for arbitrary integers m and n, we can establish if a semigroup S is an anti-inverse semigroup or not. So we have:

$$\mathcal{S}_{1,n} \subset \mathcal{A},\ \mathcal{S}_{2,n} \subset \mathcal{A}\ (n>1),\ \mathcal{S}_{m,2} \subset \mathcal{A},$$
$$\mathcal{S}_{m,m} \subset \mathcal{A},\ \mathcal{S}_{m,mq} \subset \mathcal{A}$$

and so on.

For example, if in a semigroup S hold

$$(\forall x)(\exists y)(x^{16} = y^{16} \wedge x^{16} = (xy)^{16} \wedge x^{67} = x),$$

then $S \in \mathcal{A}$.

Also, in every semigroup the following formula is true

$$(\forall x)(\exists y)(x^{62} = y^{62} \wedge x^{62} = (xy)^{62} \wedge x^{182} = x) \Rightarrow (\forall x)(x^2 = x).$$

The class $\mathcal{S}_{m,m+1}$ is not a subclass of anti-inverse semigroups for $m > 2$. This class contains all groups of order m, and so the cyclic group of order m, which is not an anti-inverse semigroup.

Beside the class $\mathcal{S}_{m,n}$, in [3], [6] is considered also the class $\mathcal{S}^*_{m,n}$, i.e. the class of semigroups for which

(2) $$(\forall x)(\exists y)(x^m = y^m \wedge yx = x^{m+1}y \wedge x^n = x).$$

An element y of a semigroup S is said to be (m,n)-anti-inverse for an element x if (1) is satisfied. Similarly, (m,n)*-anti-inverse element is defined by (2). Denote by M_a (resp. M^*_a) the set of all (m,n)-anti-inverse elements (resp. (m,n)*-anti-inverse elements) with respect to the given element $a \in S$. Then the following theorem is valid

Theorem 2.1. [3]. Let $S \in \mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$), then for every $a \in S$ and every $I^*_a \subset M^*_a$ ($I_a \subset M_a$)

$$GI^*_a := [a \cup I^*_a] \quad (GI_a := [a \cup I_a])$$

is a group.

As a consequence we have

$$S \in \mathcal{S}^*_{m,n} \Rightarrow S = \bigcup_{a \in S} GI^*_a$$
$$S \in \mathcal{S}_{m,n} \Rightarrow S = \bigcup_{a \in S} GI_a.$$

Now we shall prove the following

Theorem. Let S be a commutative semigroup. Then

$$\mathcal{S}^*_{m,n} = \mathcal{S}_{m,n}.$$

Proof. Let $S \in \mathcal{S}^*_{m,n}$ and $x \in S$ arbitrary. Then there exists $y \in S$ which is $(m,n)^*$-anti-inverse with respect to x, which implies $yx = x^{m+1}y$; hence, by the supposition on S, we have $xy = x^{m+1}y$. As $x,y \in GI_x$, where I_x is chosen so that $y \in I_x$, we have $x = x^{m+1}$, i.e. $x^m = e_x$ for all $x \in S$.

As x^{n-1} is an idempotent, the proper unity e_x of the element x and its $(m,n)^*$-anti-inverse y element are equal. By the matter of fact

$$e_x = x^{n-1} = (x^{n-1})^m = (x^m)^{n-1} = (y^m)^{n-1} =$$
$$= (y^{n-1})^m = y^{n-1} = e_y.$$

So

$$x^m = y^m = e_x, \quad x^m = x^m e_x = x^m y^m = (xy)^m$$

and $S \in \mathcal{S}_{m,n}$.

Conversely, let $S \in \mathcal{S}_{m,n}$. Then from $x^m = y^m = (xy)^m$, where y is an (m,n)-anti-inverse for x, and by the commutativity of the semigroup S, we have $y^m = x^m y^m$. Hence, by $x,y \in GI_x$, $x^m = e_x$ for all $x \in S$. In this case

$$(\forall x)(\exists y)(x^m = y^m = e_x \wedge yx = xy \wedge x^n = x).$$

As $yx = xy = e_x xy = x^m xy = x^{m+1}y$, we obtain $S \in \mathcal{S}^*_{m,n}$.

In [6] is given an algorithm for determining a basic class in the sence of Lapin for the class $\mathcal{S}^*_{m,n}$ $(m,n \in \mathbb{N})$.

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ON THE LINEAR COMPLEMENTARITY PROBLEM

D.L. Karčicka

Several important for the applications mathematical problems can be formulated as the Linear Complementarity problem:

(LCP) Find n-vectors w and z which satisfy the conditions $w=q+Mz$, $w\geq 0$, $z\geq 0$, $w^T z=0$, where the vector q and the matrix M are given.

The (LCP) has received a remarcable attention. Very often a paper on (LCP) can be found in the publications: Mathematical Programming, Linear Algebra and Its Applications, Journal of Optimization Theory and Applications. But, there is no answer on the questions of existence and uniqueness of solution, and computing solutions for any matrix M and vector q.

For certain classes of matrices, or for special q, there have been developed constructive procedures for solving the (LCP). The most of them evolved from the Complementary Pivot Method of Lemke-Howson [2] and the Principal Pivoting Method of Cottle-Dantzig [1].

Relative to (LCP) Lemke [3] defines the classes of matrices:

$\mathcal{Q}$: $M\in\mathcal{Q} \Leftrightarrow$ a solution exists for all q;

$\mathcal{K}$: $M \in \mathcal{K} \Leftrightarrow$ a solution exists for all q for which a nonnegative solution of $w = q + M\cdot z$ exists.

Obviosly, $\mathcal{Q} \subset \mathcal{K}$. In particular the $\mathcal{P}$-matrices (:positive principal minors) and more general the $\mathcal{E}$-matrices ($\forall o \neq x \geq o \ \exists i: x_i > 0$ and $(Mx)_i > o$) are $\mathcal{Q}$-matrices; the $\mathcal{Z}$-matrices ($(M)_{ij} \leq 0$ for all i and j such that $i \neq j$) and the copositive-plus matrices ($x^T M x \geq 0$ for all $x \geq o$, $(M+M^T)x = o$ if $x^T M x = o$ and $x \geq o$) are $\mathcal{K}$-matrices.

We consider the (LCP) for M of type

$$\pm((a + 1)P - E) \tag{1}$$

where a is an arbitrary real, P is a given permutation matrix of order n and E is the $n \times n$-matrix all of whose elements are unity.

Though (1) has a very special structure, it need not be a $\mathcal{K}$-matrix for any a.

For $a + 2 - n > 0$, $M \in \mathcal{K}$ [4] and in the case when a solution exists, it can be obtained performing one block pivot on a principal sub-matrix M_{II} (card I = = $r \leq n$) of type

$$\pm((a + 1)P_{II} - E_{II}) \tag{2}$$

or

$$\pm\begin{bmatrix} (a+1)P_{I'J} - E_{I'J} & -e_{I'} \\ -e_J^T & -1 \end{bmatrix} \tag{3}$$

where $I' \cup J = I$, $\text{card}(I' \cap J) = r - 2$ and $e_{I'}$ and e_J are r-1-vectors all of whose elements are unity. As

$$\frac{1}{a+1}\left[P_{II}^T + \frac{1}{a+1-r}E_{II}\right] \quad \text{is the inverse of (2)}$$

and

$$\frac{1}{a+1}\begin{bmatrix} P_{I'J}^T & -e_J \\ -e_{I'}^T & -(a+1-r) \end{bmatrix} \quad \text{is the inverse of (3)}$$

the solution $(\bar{w};\bar{z}) = (o,\bar{w}_{\underline{I}};\bar{z}_I,o)$ $(I \cup \underline{I} = \{1,\ldots,n\}$, $I \cap \underline{I} = \emptyset)$ is computed applying the formulas

$$\bar{z}_I = -M_{II}^{-1}q_I, \qquad \bar{w}_I = o$$
$$\bar{w}_{\underline{I}} = q_{\underline{I}} + M_{\underline{I}I}\bar{z}_I, \qquad \bar{z}_{\underline{I}} = o. \tag{4}$$

For $M = (a+1)P-E$ $(M=E-(a+1)P)$ let

$$q_r = \max_{1\leq j\leq n}\{q_j\} \qquad (q_r = \min_{1\leq j\leq n}\{q_j\})$$

$$\ell = \pi^{-1}(r), \quad s = \sum_{j=1}^{n} q_j + (a+1-n)q_r,$$

where $\pi(j) = i_j \Leftrightarrow (M)_{i_j j} = \pm a$.

The (LCP) is infeasible if $s < 0$ and $n-2<a\leq n-1$ ($s < 0$ and $a \geq n-1$). The pair of vectors $(\bar{w};\bar{z}) = (o;-M^{-1}q)$ is a solution if $s < 0$ and $a>n-1$, or $s \geq 0$ and $n-2<a<n-1$ (if $s < 0$ and $n-2<a<n-1$, or $s \geq 0$ and $a>n-1$). Otherwise, the set of indices I corresponding to the solution (4) can be obtained applying the following algorithm:

Step 0. Initialize $\nu = 0$, $I^\nu = \{1,\ldots,n\}$, $r_\nu = r$, $\ell_\nu = \ell$, $s_\nu = s$ and go to step 1.

Step 1. Set $I^{\nu+1} = I^\nu - \{\ell_\nu\}$ and test $r_\nu \neq \ell_\nu$ (test $r = \ell_\nu$).

1.1. If yes, then $I = I^{\nu+1}$ and M_{II} is of type (3) (or type (2)); stop!

1.2. If no, then go to step 2.

Step 2. Set $\nu = \nu+1$, find $q_{r_\nu} = \max_{j\in I^\nu}\{q_j\}$, $\ell_\nu = \pi^{-1}(r_\nu)$ $(\ell_\nu = \pi^{-1}(\ell_{\nu-1}))$,

$$s_\nu = \sum_{j\in I^\nu} q_j + (a+1-n+\nu)q_{r_\nu} \quad (s_\nu = \sum_{j\in I^\nu} q_j + (a+1-n+\nu)q_r)$$

and test $s_\nu \leq 0$:

2.1. If yes, then $I = I^{\nu}$ and M_{II} is of type (2) (of type (3)); stop!

2.2. If no, go to step 1.

A modification of the algorithm can be used for solving:

a) The parametric (LCP) [5], [6]

$w = (q+\alpha p) + Mz, \quad v \geq o, z \geq o, w^T z = 0,$

α-parameter;

b) The dual pair of LP-problems [7]

$\min\{pz | q + Mz \geq o, z \geq o\}, \quad \max\{-vq \mid p-vM \geq o, v \geq o\};$

c) The quadratic program [8]

$\min\{x^T M_1^T M_1 x \mid M_2 x \geq c\}.$

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BI-IDEAL SEMIGROUPS

B. Trpenovski

We call a semigroup S a bi-ideal semigroup iff all subsemigroups of S are bi-ideals in S, i.e. $B \subseteq S$, $B^2 \subseteq B \Rightarrow BSB \subseteq B$. Bi-ideal semigroups were introduced in [6] in an analogous way as the left-ideal semigroups were introduced and studied in [3] and [7]. It seems, however, that the way the structure of left-ideal semigroups is described in [3] and [7] is not appropriate in the case of bi-ideal semigroups. So, we explore here the idea from [2] to give a structural description for bi-ideal semigroups. First, let us quote some of the results from [6]:

<u>Theorem 1.</u> Let S be a bi-ideal semigroup. Then the following hold:

(i) $(\forall a \in S)$ $aSa \subseteq \langle a \rangle$, where $\langle a \rangle$ is the cyclic subsemigroup of S generated by a;

(ii) S is periodic and $|\langle a \rangle| \leq 5$ for all $a \in S$;

(iii) The set E of all the idempotents of S is a rectangular band;

(iv) $(\forall e \in E)(\forall x \in S)$ $xe, ex \in E$. ∎

In what follows we suppose S to be a bi-ideal semigroup.

Let us put $P = S\setminus E$, where E is as in Theorem 1. We shall establish some properties about P and S.

a) It is easily seen that P is a partial semigroup, i.e. $(\forall x,y,z \in P)$ if one of the elements $(xy)z$ and $x(yz)$ belongs to P, then $(xy)z, x(yz) \in P$ and $(xy)z = x(yz)$. ■

From Theorem 1 it follows that

b) $(\forall x \in P)(\exists m \in N)$, where N is the set of positive integers, such that $x^m \notin P$. (In fact, $(\forall x \in P)$ $x^5 \notin P$). Because of this property we may call P a periodic partial semigroup. ■

A subset R of a partial semigroup Q is said to be a partial subsemigroup of Q iff [$x,y \in R$ and $xy \in Q$, then $xy \in R$]. A partial subsemigroup R of a partial semigroup Q is said to be a bi-ideal in Q iff $x,y \in R$, $xqy \in Q$, $q \in Q$, implies $xqy \in R$. If all partial subsemigroups of a partial semigroup Q are bi-ideals in Q, then we call Q a partial bi-ideal semigroup. We can show, now,

c) P is a partial bi-ideal semigroup.

Really, if B is a partial subsemigroup of P, $x,y \in B$ and $xpy \in P$, $p \in P$, then $B^* = \langle B\rangle$ in S is a bi-ideal in S and therefore $xpy \in B^*$. But, from $B^*\setminus B \subseteq E$ and $P \cap E = \emptyset$ it follows that $xpy \in B$. ■

Let e_x be the idempotent in $\langle x\rangle$ and let us put $\phi(x) = e_x$. Then,

d) $\phi: P \to E$ is a homomorphism.

If $xy=z$, $x,y,z \in P$, then $zx=xyx \in \langle x\rangle$ in S, i.e. $zx = x^k$, $k \in \{1,2,3,4,5\}$. Let $x^m=e_x$. From $zx=x^k$ it follows that $zx^m=x^{m+k-1}$, i.e. $ze_x=e_x$, since $x^{m+k-1} = e_x x^{k-1}$ is a idempotent (th. 1 (iv)) which belongs to $\langle x\rangle$. Now, $z^2e_x=ze_x=e_x$, $z^3e_x=e_x$ and so on, so that $e_ze_x=e_x$. Similarly we have $e_ye_x=e_x$ and then $e_xe_y=e_ze_xe_ye_z \in \langle e_z\rangle$ so that $e_xe_y=e_z$ which means that $\phi(x)\phi(y) = \phi(xy)$. ■

Let $x,y \in S$ and $xy \in E$. Then in a similar way as above we can prove that $ee_x=e_x$ and $e_ye=e_y$ where $e=xy$. Now, $xy=e=ee_xe_ye=e_xe_y=\phi(x)\phi(y)$:

e) $x,y \in S$, $xy \in E \Longrightarrow xy = \phi(x)\phi(y)$. ■

If we put $\phi(e)=e$ for all $e \in E$, then from the definition of ϕ and e) it follows that we can extend $\phi: S \to E$ to be an epimorphism.

Conversely, assume that P is a periodic partial bi-ideal semigroup, E a rectangular band, $P \cap E = \emptyset$ and $\phi: P \to E$ a homomorphism. By putting $\phi(e)=e$ for all $e \in E$, we can consider ϕ as a mapping from $S = P \cup E$ onto E such that $\phi_{|P}$ is a homomorphism. We define an operation in S by

$$xy = \begin{cases} xy \text{ as in P if } x,y \in P \text{ and } xy \text{ is defined in P,} \\ \phi(x)\phi(y) \text{ otherwise.} \end{cases}$$

Let us show that S is a semigroup. Let $x,y,z \in S$. We consider the following three cases:

(i) If one of $(xy)z$ and $x(yz)$ belongs to P, then as P is a partial semigroup, we have that $(xy)z$, $x(yz) \in P$ and $(xy)z = x(yz)$.

(ii) If both, xy and yz are not defined in P, then $(x,y)z$, $x(yz) \in E$ and by the definition of the opperation in S and the associativity in E we have that

$$(xy)z=[\phi(x)\phi(y)]\phi(z)=\phi(x)[\phi(y)\phi(z)]=x(yz).$$

(iii) Finally, if at least one of xy, yz (for instance xy) is defined in P but neither of $(xy)z$ and $x(yz)$ is defined in P, then

$$(xy)z=\phi(xy)\phi(z)=(\phi\text{-homomorphism})=$$
$$=[\phi(x)\phi(y)]\phi(z)=(\text{associativity in } E)=$$
$$=\phi(x)[\phi(y)\phi(z)]=(\text{definition of } \phi,$$
$$\text{or } \phi\text{-homomorphism})=\phi(x)\phi(yz)=x(yz).$$

Denote the semigroup just constructed by $S=(P,E,\phi)$. We shall prove, now, that $S=(P,E,\phi)$ is a bi-ideal semigroup.

Let B be a subsemigroup of S, $x,y\in B$ and $s\in S$. It is clear that $B^*=B\setminus E$ is a partial subsemigroup of P. So, if $xsy\in P$, then $xsy\in B^*\subseteq B$ since P is a partial bi-ideal semigroup. Let xsy is not defined in P. If $xy\in B^*$, then xy is not defined in P and

$$xsy=\phi(x)\phi(s)\phi(y)=(E \text{ is a rectangular band})=$$
$$=\phi(x)\phi(y)=xy\in B.$$

Finally, if $xy\in P$, then $xy\in B^*$ and, because of the periodicity of P, $(xy)^k\in E$ for some $k\in N$. Let $(xy)^k=e$. We have that $e\in B\setminus B^*$ and, since ϕ is a homomorphism, $\phi(x)\phi(y)=\phi(xy)\in E$. Now,

$$\phi(xy)=[\phi(xy)]^k=\phi[(xy)^k]=e.$$

So, again we have that

$$xsy=\phi(x)\phi(y)=\phi(xy)=e\in B,$$

which proves that B is a bi-ideal of S.

In summary, we have proved the following

Theorem 2. A semigroup S is a bi-ideal semigroup iff $S=(P,E,\phi)$ where P is a periodic partial bi-ideal semigroup, E a rectangular band, $P\cap E=\emptyset$ and $\phi:P\to E$ a homomorphism.■

At the end, using Theorem 2, let us quote some examples of bi-ideal semigroups.

Examples

1) Every rectangular band is a bi-ideal semigroup.

2) Let A be a nonempty set, E-rectatgular band and $\phi:A\to E$ any mapping. Then $S=A\cup E$ is a bi-ideal semigroup with an operation defined as follows:

$$xy=\begin{cases}\phi(x)\phi(y) & \text{if } x,y\in A\\ xy & \text{if } x,y\in E\\ \phi(x)y & \text{if } x\in A, y\in E\\ x\phi(y) & \text{if } x\in E, y\in A.\end{cases}$$

3) Let E be a rectangular band and B_k a partial semigroup defined as follows: (i) $x,y\in B_k$, $x\neq y \Longrightarrow xy$ is not defined in B_k; (ii) $x\in B_k \Longrightarrow x^2\in B_k$ (k=2), $x^2,x^3\in B_k$ (k=3), $x^2,x^3,x^4\in B_k$ (k=4). Further, let $\phi:B_k\to E$ be a mapping such that, if $x^k\in B_k$, then $\phi(x^k)=\phi(x)$. Let us extend ϕ to a mapping from $S=B_k\cup E$ onto E by $\phi(e)=e$ for all $e\in E$ If we define an operation in S by $xy=\phi(x)\phi(y)$, then S will be a bi-ideal semigroup.

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n-DIMENSIONAL SEMINETS AND PARTIAL n-QUASIGROUPS

Kiril Stojmenovski

It is shown that there is an equivalence between the theories of n-dimensional seminets and regular partial n-quasigroups.

0. First we give some necessary definitions.

Let Q be a non-empty set, $D \subseteq Q^n$, and A be a mapping from D into Q such that:

$$A(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n) = \\ = A(a_1,\dots,a_{i-1},y,a_{i+1},\dots,a_n) \implies x = y \quad (0.1)$$

for each $i \in \{1,2,\dots n\}$. Then (Q,A) is called a partial n-quasigroup with domain D. If $a \in Q$ and $i \in \{1,2,\dots,n+1\}$, then $b_{a,i}$ is a subset of Q^{n+1} defined by:

$$b_{a,i} = \{(x_1,\dots,x_{n+1}) \mid x_i=a, \ A(x_1,\dots,x_n)=x_{n+1}\}. \quad (0.2)$$

A partial n-quasigroup (Q,A) is said to be regular if the following statement is satisfied:

(R) The domain D of (Q,A) is non-empty and if $A(a_1,\dots,a_n) = a_{n+1}$, then the sets $b_{a_1,1},\dots,b_{a_{n+1},n+1}$ are distinct.

Let (Q,A) and (Q′,A′) be partial n-quasigroups with domains D and D´ respectively and let $\alpha_1,\dots,\alpha_{n+1}$ be partial bijections from Q into Q´. (By a partial bijection α from Q into Q´ we mean a bijection from a subset of Q into a subset of Q´). We say that $(\alpha_1,\dots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q´,A´) if:

$$(x_1,\dots,x_n)\in D \Leftrightarrow (\alpha_1(x_1),\dots,\alpha_n(x_n))\in D',$$

$$A(D) \text{ is in the domain of } \alpha_{n+1}, \tag{0.3}$$

$$\alpha_{n+1}(A(x_1,\dots,x_n)) = A'(\alpha_1(x_1),\dots,\alpha_n(x_n)),$$

for every $(x_1,\dots,x_n)\in D$. In that case we say that (Q,A) and (Q´,A´) are isotopic. It is clear that if one of (Q,A), (Q´,A´) is regular, the another one is also regular.

Let P be a nonempty set and let $B_1,\dots,B_{n+1}$ $(n \geq 2)$ be nonempty mutually disjoint collections of subsets of P. The elements of P are called points and the elements of the sets B_ν are called blocks. We say that $(P;B_1,\dots,B_{n+1})$ constitutes a structure of n-dimensional seminet (or n-seminet) iff the following two statements are satisfied:

(SN)(i) For every point $p\in P$ there exists a unique sequence of blocks $b_1,\dots,b_{n+1}$ $(b_\nu \in B_\nu)$ such that $\{p\} = b_1\cap\dots\cap b_{n+1}$.

(ii) For every $i\in\{1,2,\dots,n+1\}$ and every sequence of blocks $b_1,\dots,b_{i-1}, b_{i+1},\dots,b_{n+1}$ such that $b_\nu\in B_\nu$, the set $b_1\cap\dots\cap b_{i-1}\cap b_{i+1}\cap\dots\cap b_{n+1}$ contains at most one point.

1. Here we will show that every regular partial n-quasigroup induces an n-seminet.

Let (Q,A) be a regular partial n-quasigroup, and let P (the set of points) be defined by:

$$P = \{(x_1,\dots,x_{n+1}) \mid A(x_1,\dots,x_n) = x_{n+1}\}. \tag{1.1}$$

Every non-empty set:

$$b_{x,i} = \{(x_1,\dots,x_{n+1})\in P \mid x_i = x\} \tag{1.2}$$

is called a block, and the sequence of blocks $B_1,\dots,B_{n+1}$ is defined by:

$$B_i = \{b_{x,i} \mid b_{x,i} \neq \emptyset,\ x\in Q\} \tag{1.3}$$

From (R) if follows that $P\neq\emptyset$, and that $B_1,\dots,B_{n+1}$ are nonempty and disjoint.

If $p = (a_1,\dots,a_{n+1})\in P$, then $b_{a_1,1},\dots,b_{a_{n+1},n+1}$ is the unique sequence such that $b_{a_\nu,\nu}\in B_\nu$ and $\{p\} = b_{a_1,1}\cap\dots\cap b_{a_{n+1},n+1}$. If $b_{a_\nu,\nu}\in B_\nu$ $(\nu = 1,2,\dots,n)$, then $b_{a_1,1}\cap\dots\cap b_{a_n} = (a_1,\dots,a_{n+1})$, where $a_{n+1} = A(a_1\dots a_n)$. Also if $b_{a_\nu,\nu}\in B_\nu$ $(\nu = 1,\dots i-1, i+1,\dots,n+1, i<n+1)$, then

$$b_{a_1,1}\cap\dots b_{a_{i-1},i-1}\cap b_{a_{i+1},i+1}\cap\dots b_{a_{n+1},n+1} = \{p,q\}$$

implies that

$$p = (a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_{n+1}),$$

$$q = (a_1,\dots,a_{i-1}y,a_{i+1},\dots,a_{n+1}),$$

$$a_{n+1} = A(a_1\dots a_{i-1},x,a_{i+1}\dots a_n) =$$

$$= A(a_1\dots a_{i-1}ya_{i+1}\dots a_n);$$

whence by (0.1) we obtain that $x = y$, i.e. $p = q$.

This completes the proof that $(P;B_1,\dots,B_{n+1})$ is an n-seminet.

If (Q,A) and (Q',A') are isotopic, then it can be easily shown that the corresponding n-seminets are isomorphic.

Namely, if $(\alpha_1,\ldots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q',A'), then the mapping $\phi\colon P \to P'$ defined by:

$$\phi : (a_1,\ldots,a_{n+1}) \to (\alpha_1(a_1),\ldots,\alpha_{n+1}(a_{n+1}))$$

induces an isomorphism from $(P; B_1,\ldots,B_{n+1})$ into $(P'; B_1',\ldots,B_{n+1}')$.

2. Now we will show that every n-seminet can be coordinatized by a partial regular n-quasigroup.

Let $(P;B_1,\ldots,B_{n+1})$ be an n-seminet such that $m = \max\{|B_i|\,|\,i = 1,2,\ldots,n+1\}$ and let Q be a set with $|Q| = m$. If $f_i\colon B_i \to Q$ is an injection for each $i = 1,\ldots,n+1$, then a partial n-ary operation A can be defined in Q by:

$$A(q_1,\ldots,q_n) = q_{n+1} \iff (\exists b_1,\ldots,b_{n+1};\ b_\nu \in B_\nu)\ b_1 \cap \ldots \cap b_{n+1} \neq \emptyset,\ q_i = f_i(b_i)$$

From (SN) we obtain first that A is a partial n-quasigroup, and it is regular as well, for $B_1,\ldots,B_{n+1}$ are disjoint.

Certainly, (Q,A) depends on the injections $f_1,\ldots,f_{n+1}$ and the set Q. Assume that the sequence of injections $f_1\colon B_i \to Q$ induces the partial n-quasigroup (Q,A) and the injections $f_1'\colon B_i \to Q'$ – the n-quasigroup (Q',A'). If we put $\alpha_\nu = f_\nu' f_\nu^{-1}$, then we get a partial bijection α_ν from Q into Q'. Then $(\alpha_1,\ldots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q',A').

Example. Given a 2-seminet as in Fig. 1 with:

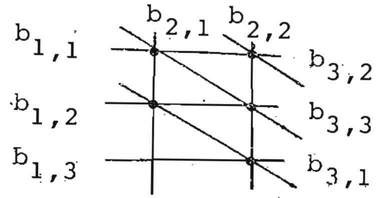

Fig. 1.

$$B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}\},$$
$$B_2 = \{b_{2,1}, b_{2,2}\},$$
$$B_3 = \{b_{3,1}, b_{3,2}, b_{3,3}\},$$
$$\max\{|B_i|\,|\,i \in \{1,2,3\}\} = 3,$$
$$Q = \{x_1,x_2,x_3\}.$$

If

$$f_1 = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ x_3 & x_2 & x_1 \end{pmatrix}; \quad f_2 = \begin{pmatrix} b_{2,1} & b_{2,2} \\ x_3 & x_2 \end{pmatrix};$$

$$f_3 = \begin{pmatrix} b_{3,1} & b_{3,2} & b_{3,3} \\ x_1 & x_3 & x_2 \end{pmatrix}; \quad g_1 = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ x_1 & x_3 & x_2 \end{pmatrix};$$

$$g_2 = \begin{pmatrix} b_{2,1} & b_{2,2} \\ x_2 & x_1 \end{pmatrix}; \quad g_3 = \begin{pmatrix} b_{3,1} & b_{3,2} & b_{3,3} \\ x_2 & x_3 & x_1 \end{pmatrix},$$

then we obtain partial 2-quasigroups (Q,A) and (Q,B):

A	x_1	x_2	x_3
x_1	x_1		
x_2	x_2		x_1
x_3	x_3		x_2

B	x_1	x_2	x_3
x_1	x_3	x_1	
x_2	x_2		
x_3	x_1	x_2	

If

$$\alpha_1 = f_1 g_1^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}; \quad \alpha_2 = f_2 g_2^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_1 & x_3 \end{pmatrix};$$

$$\alpha_3 = f_3 g_3^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix},$$

then the ordered triple $(\alpha_1, \alpha_2, \alpha_3)$ is an isotopy of (Q,B) and (Q,A).

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ON A CLASS OF NORMAL SEMIGROUPS

P. Kržovski

A semigroup S is called normal if $xS = Sx$ for all elements x of S (S.Schwarz [1]). It is considered in this note the class of normal semigroups with the property $Sx = Sx^2$ for all x of S. Two characterizations for the semigroups of this class are obtained here.

1°. If a semigroup S is normal and regular, then $Sx = Sx^2$ for all x of S.

Proof. If $x \in S$ and $x = xyx$, then

$$Sx = Sxyx \subseteq SxSx = S^2x^2 \subseteq Sx^2 = Sx.$$

The following example shows that the converse does not hold: If S is a semigroup such that $|S| > 1$ and $|S^2| = 1$, then obviously $Sx = xS = Sx^2$ and S is not regular.

2°. A semigroup S is normal and has the property $Sx = Sx^2$ for all $x \in S$ if and only if S is an inflation of a semilattice of groups.

Proof. Let S be a normal semigroup with the property $Sx = Sx^2$ for all $x \in S$. Denote by T the set of all regular elements of S. We shall prove that $T = S^2$. If $z \in S^2$, then there exist $x,y,u,v,s,t \in S$ such that

$$z = xy = uy^2 = uy^2v = xyv = (xy)^2s = xytxy,$$

which means that $z \in T$, i.e. $S^2 \subseteq T$. The inclusion $T \subseteq S^2$ is obvious. Now we shall prove that T is normal. If $x, y \in T$, then there exist $s, u \in S$ such that $xy = sx = sxusx \in S^2x = Tx$ and this implies that $xT \subseteq Tx$. By symmetry $Tx \subseteq xT$ and thus the semigroup $S^2 = T$ is regular and normal. According to [4], S^2 is a semilattice of groups. We note that idempotents of S are in the centre of S [1], and thus the set of idempotents E is a subsemigroup of $S^2 = T$.

Define a transformation ϕ of S as follows. If $x \in S$, then $x^2 \in S^2$ and thus there exists an idempotent e such that $x^2 \in G_e$. Then $\phi(x) = xe$ defines a transformation of S.

If $x, y \in S$ and $x^2 \in G_e$, $y^2 \in G_f$, then:

$$(xy)^2ef = xyxyef = xyxeyf = (xy)^2$$

and thus $(xy)^2 \in G_{ef}$, i.e.

$$\phi(xy) = xyef = xe \cdot yf = \phi(x)\phi(y).$$

Moreover, there exist $u, v \in S$ such that

$$\phi(xy) = xyef = x(ye)f = x(uy^2)f =$$
$$= xuy^2 = xye = vx^2e = vx^2 = xy.$$

Therefore ϕ is an endomorphism of S which fixes the elements of S^2 and this implies that S is an inflation of S^2.

Conversely, assume that T is a semilattice of groups, and S is an inflation of T. Then, clearly, T is a normal semigroup such that $Tt = Tt^2$ for each $t \in T$, and this implies that S is also a normal semigroup satisfying the equality $Sx = Sx^2$ for every x of S.

3°. Let S be a normal semigroup. The following statements are equivalent:

(i) $Sx = Sx^2$ for all $x \in S$;

(ii) $N(x) = \{y \in S \mid Sx \subseteq Sy\}$ for all $x \in S$.

Proof. (i) $\Longrightarrow$ (ii). First we shall prove that

$$F = \{y \in S \mid Sx \subseteq Sy\}$$

is a filter which contains the element x.

Let $y, z \in F$. Then $Sx \subseteq Sy$ and $Sx \subseteq Sz$ and since

$$Sx = Sx^2 = Sx^4 = Sxx^2x \subseteq SxSx \subseteq SySz = S^2yz \subseteq Syz,$$

it follows that $yz \in F$.

Conversely, if $yz \in F$, then $Sx \subseteq Syz \subseteq Sz$ and $Sx \subseteq Syz = ySz \subseteq yS = Sy$ which means that $y, z \in F$, i.e. F is a filter. Since $Sx \subseteq Sx$, it follows that $x \in F$ and this implies that $N(x) \subseteq F$. To show the inclusion $F \subseteq N(x)$ we use II.2.10 of [3]. If $y \in F$, then $x^2 \in Sx \subseteq Sy \subseteq J(y)$. Since $x^2 \in N_1(x) \cap J(y)$, we get $y \in N_2(x) \subseteq N(x)$ and so $F \subseteq N(x)$. Hence $F = N(x)$.

(ii) $\Longrightarrow$ (i) Obviosly $Sx^2 \subseteq Sx$ for any $x \in S$. Since $x^2 \in N(x)$, it follows that $Sx \subseteq Sx^2$. Therefore $Sx = Sx^2$.

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ЗА ЕДНА КЛАСА НОРМАЛНИ ПОЛУГРУПИ

П. Кржовски

Во оваа статија се разгледани нормалните полугрупи ($xS=Sx$ за секој $x\in S$) кои го задоволуваат условот $Sx=Sx^2$.

Ако полугрупата S е нормална и регуларна, тогаш $Sx=xS=Sx^2$. Меѓутоа обратното не важи. На пример, ако S е полугрупа таква што $|S|>1$ и $|S^2|=1$, тогаш $Sx=xS=Sx^2$, но S не е регуларна (1^o), со што покажуваме дека оваа класа е поширока отколку кога S е полумрежа од групи. За класата полугрупи $Sx=xS=Sx^2$ добиваме две карактеристики:

S е нормална полугрупа со својството $Sx=Sx^2$ ако и само ако S е инфлација на полумрежа од групи (2^o).

Ако S е нормална, тогаш следниве искази се еквивалентни:

(i) за секој $x\in S$, $Sx=Sx^2$

(ii) за секој $x\in S$, $N(x)=\{y\in S \mid Sx\subseteq Sy\}$ (3^o).

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ON QUASIVARIETIES OF GENERALIZED SUBALGEBRAS

S. Markovski

Throughout this paper we shall use the usual notations and notions of the theory of models, assuming that all languages are first order languages without relational symbols different from $\equiv$, where $\equiv$ denotes the equality symbol in the given language. The aim of this paper is to give a generalization of Theorem (M), by using a generalization of Theorem (L).

<u>THEOREM (M)</u> ([2], p. 274). Let $L_1\subseteq L_2$ be two languages and let $\Sigma_1(\Sigma_2)$ be a set of L_1-quasiidentities (L_2-quasiidentities). Then the class $\mathbb{K}$ of all L_1-algebras $\mathcal{A}\in \underline{Mod}\Sigma_1$, for which there exists an L_2-algebra $\mathcal{B}\in\underline{Mod}\Sigma_2$ such that $\mathcal{B}$ is an L_2-extension of $\mathcal{A}$, is a quasivariety. □

<u>THEOREM (L)</u> ([1]). Let $L_1\subseteq L_2$ be two languages and let Σ be a set of L_2-formulas, let $\mathcal{A}$ be an L_1-algebra. Then there exists an L_2-extension $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}\in\underline{Mod}\Sigma$ iff $\mathcal{A}$ is a model of the set of all open L_1-formulas, which are theorems in the theory defined by Σ. □

Here by a quasiidentity we mean a formula of the form $\neg\phi_1\vee\dots\vee\neg\phi_k\vee\phi$, where $\phi,\phi_1,\dots,\phi_k$ ($k\geq 0$) are atomic formulas (or identities), i.e. formulas of the form $\xi \equiv \eta$, where $\xi, \eta \in \text{Term}(L)$. A quasivariety is a class $\mathbb{K}$ of L-algebras such that $\mathbb{K} = \underline{\text{Mod}}\,\Sigma$, where Σ is a set of quasiidentities.

1. Generalized subalgebras. Let L_1 and L_2 be two languages such that $L_1 \cap L_2$ does not contain operational symbols. (This assumption is only for technical reasons, as can be seen from what follows.) For each n-ary operational symbol $f \in L_1$ let a term $f^\wedge \in \text{Term}(L_2)$ be given such that $f^\wedge = f^\wedge(x_1, \ldots, x_n)$ contains no more than n distinct variables. Then an L_1-algebra $\mathcal{A}$ is said to be a $^\wedge$-subalgebra (or generalized subalgebra) of an L_2-algebra $\mathcal{B}$ iff $|\mathcal{A}| \subseteq |\mathcal{B}|$ and

$$f_{\mathcal{A}}(a_1, \ldots, a_n) = f^\wedge_{\mathcal{B}}(a_1, \ldots, a_n) \qquad (1.1$$

for all $a_1, \ldots, a_n \in |\mathcal{A}|$, $f \in L_1$ ($|f| = n$, $n = 0, 1, 2, \ldots$).

Let $\mathcal{A}$ be an L_1-algebra. We are asking under what circumstances there exists an L_2-algebra $\mathcal{B}$ belonging to a class of L_2-algebras $\mathbb{K}$, such that $\mathcal{A}$ is a generalized subalgebra of $\mathcal{B}$ (for a given $^\wedge$). We shall give here an answer which is a generalization of Theorem (L).

Let the languages L_1, L_2 and the mapping $^\wedge$ be as above. Let Σ be a set of L_2-formulas and

$$\Sigma' = \Sigma \cup \{f x_1 \ldots x_n \equiv f^\wedge(x_1, \ldots, x_n) \mid f \in L_1\}.$$

THEOREM 1 ([5], [7]). An L_1-algebra $\mathcal{A}$ is a $^\wedge$-subalgebra of some L_2-algebra $\mathcal{B} \in \underline{\text{Mod}}\,\Sigma$ iff $\mathcal{A}$ satisfies all open formulas, which are theorems in the theory defined by the set of formulas Σ'.

Proof. Let $\mathcal{A}$ be a $^\wedge$-subalgebra of an L_2-algebra $\mathcal{B} \in \underline{\text{Mod}}\,\Sigma$. Form an expansion $\mathcal{C}$ of $\mathcal{B}$ for the language $L_1 \cup L_2$ by putting $f \in L_1 \Rightarrow f_{\mathcal{C}}(b_1, \ldots, b_n) = f^\wedge_{\mathcal{B}}(b_1, \ldots, b_n)$, for all $b_1, \ldots, b_n \in |\mathcal{B}|$, $f \in L_1$ ($|f| = n$, $n = 0, 1, 2, \ldots$). Then $\mathcal{A}$ is subalgebra of $\mathcal{C}|L_1$, $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, and so every open L_1-formula, satisfied by $\mathcal{C}$, is satisfied by $\mathcal{A}$ too.

Conversely, suppose that $\mathcal{A}$ satisfies all open L_1-formulas which are theorems in the theory defined by Σ'. Then, by Theorem (L), there exists an $L_1 \cup L_2$-algebra $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, such that $\mathcal{A}$ is an L_1-subalgebra of $\mathcal{C}$. Consider the resfriction $\mathcal{B}|L_2$. Since $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, we have $\mathcal{B} \in \underline{\text{Mod}}\,\Sigma$ and, furthermore, for all $f \in L_1$, $a_1, \ldots, a_n \in |\mathcal{A}|$,

$$f_{\mathcal{A}}(a_1, \ldots, a_n) = f_{\mathcal{C}}(a_1, \ldots, a_n)$$

(as $\mathcal{A}$ is an L_1-subalgebra of $\mathcal{C}$),

$$f_{\mathcal{C}}(a_1, \ldots, a_n) = f^\wedge_{\mathcal{B}}(a_1, \ldots, a_n)$$

(as $\mathcal{B} = \mathcal{C}|L_2$)

So, (1.1) is satisfied. □

2. Malcev's theorem for generalized subalgebras. Now we shall give a generalization of Malcev's theorem in the case when generalized subalgebras are considered. Since the usual notion of subalgebra can be obtained as a special kind of generalized subalgebra, the proof given below is another proof of Theorem (M) too (and more elementary, in author's opinion).

THEOREM 2. Let L_1, L_2 be two languages and $^\wedge$ be a given mapping (defined as above), and let Σ_1 (Σ_2) be a set of L_1-quasiidentities (L_2-quasiidentities). Then the class $\mathbb{K}$, consisting of all L_1-algebras $\mathcal{A} \in \underline{\text{Mod}}\,\Sigma_1$ which are $^\wedge$-subalgebras of algebras belonging to the $\underline{\text{Mod}}\,\Sigma_2$, is a quasuvariety.

Proof. Denote by Σ_o the set of all open L_1-formulas as in the proof of Theorem 1. It suffices to prove that $\underline{\text{Mod}}\,\Sigma_o = \underline{\text{Mod}}\,\Sigma_o'$, where Σ_o' consists of all quasiidentities from Σ_o.

We may assume that Σ_o contains only formulas of the following form:

$$\neg\phi_1 \vee \dots \vee \neg\phi_n \qquad (2.1)$$

$$\phi_1 \vee \dots \vee \phi_m \qquad (2.2)$$

$$\neg\phi_1 \vee \dots \vee \neg\phi_k \vee \phi_{k+1} \vee \dots \vee \phi_{k+p} \quad (m,n,k,p > 0) \qquad (2.3)$$

where ϕ_i are atomic formulas (identities).

Let $\Sigma_3 = \Sigma_1 \cup \Sigma_2 \cup \{fx_1 \dots x_n \equiv f^\wedge(x_1,\dots,x_n) \mid f \in L_1\}$. Then Σ_3 is a set of quasiidentities, and

$$\mathbb{M} = (\underline{\mathrm{Mod}}\,\Sigma_3)\mid_{L_1} \subseteq \underline{\mathrm{Mod}}\,\Sigma_o.$$

Since each one element L_1-algebra belongs to $\mathbb{M}$, we get that Σ_o does not contain formulas of the forms (2.1). The free $L_1 \cup L_2$-algebras belong to the class $\underline{\mathrm{Mod}}\Sigma_3$, and so every formula of the form (2.2) is equivalent to a formula of the form ϕ_i, for some $i{:}1 \leq i \leq m$. As $\underline{\mathrm{Mod}}\Sigma_3$ is closed under direct products, we have that every formula of the form (2.3) is equivalent to a formula of the form $\neg\phi_1 \vee \dots \vee \neg\phi_k \vee \phi_{k+j}$, for some $j{:}1 \leq j \leq p$. □

A simpler formulation of Theorem 2 is given by:

THEOREM 2′. The class $\mathbb{K}$ of all ^-subalgebras of algebras belonging to a quasivariety $\mathbb{Q}$ is a quasivariety. □

The folloving problems arise about the class $\mathbb{K}$:

1) give a convenient description of the quasivariety $\mathbb{K}$;

2) under what conditions $\mathbb{K}$ is a variety, when $\mathbb{Q}$ is a variety.

Remark. If the languages L_1 and L_2 contain relational symbols, then Theorem 1 still holds. In this case, for a relational symbols p, we put p^ to be an L_2-formula with no more than n free variables ($|p| = n$). Then, in addition of (1.1), we put

$$P_{\mathcal{A}}(a_1,\dots,a_n) = T \text{ iff } P^{\wedge}_{\mathcal{B}}(a_1,\dots,a_n) = T.$$

Furthermore, if p^ is an atomic formula, then Theorem 2 holds too (with corresponding preformulations, of course).

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POLYADIC SUBSEMIGROUPS OF SEMIGROUPS

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This work is an attempt to systematize a part of the results on generalized subsemigroups of semigroups obtained up to now mainly by the authors of this paper. The majority of the results is already published and we usually quote the paper where the corresponding result is published. The results published here for the first time are 3.11° and the most part of §5 and §6. We give also proofs to some known results in §4 with a purpose to illustrate the so called indirect method, i.e. the usefulness of Post Coset Theorem for investigation of n-groups by using binary groups.

§1. Universal enveloping semigroups

Let $\underline{A} = (A; F)$ be an algebra with a carrier A and a nonempty set of finitary operators, $F = F_2 \cup F_3 \cup \dots \cup F_n \cup \dots$, where F_n consists of n-ary operators belonging to F. Denote by Λ the set of "semigroup defining relations" $a = a_1 a_2 \dots a_n$, where $a = fa_1 \dots a_n$ in $\underline{A}$. Then the semigroup $A^\wedge$ with a presentation $\langle A; \Lambda \rangle$ (in the variety of semigroups) is called a *universal enveloping semigroup* for the algebra A. If

$u = a_1 \dots a_k$ ($a_\nu \in A$) is a word on A, then by $u^\wedge$ is denoted the element of $A^\wedge$ determined by u. Define a mapping $\lambda : A \to A^\wedge$ by $\lambda : a \to a^\wedge$. Then we have the following universal property of $A^\wedge$.

1.1°. If $\underline{A} = (A;F)$ and $\underline{A}' = (A';F)$ are F-algebras and $\phi : \underline{A} \to \underline{A}'$ a homomorphism, then there exists a unique homomorphism $\phi^\wedge : A^\wedge \to A'^\wedge$ such that $\lambda'\phi = \phi^\wedge\lambda$.

(We note that if ϕ is an epimorphism or isomorphism, then $\phi^\wedge$ has the corresponding property but it may happen ϕ to be a monomorphism and $\phi^\wedge$ not to be; [7].)

We say that $\underline{A}$ is an F-semigroup if there exists a semigroup S such that $A \subseteq S$ and

$$fa_1 a_2 \dots a_n = a_1 a_2 \dots a_n$$

for every n-ary operator $f \in F$ and $a_1, a_2, \dots, a_n \in A$. If, in addition, S is generated by the set A, then S is called a covering semigroup of $\underline{A}$.

1.2°. $\underline{A}$ is an F-semigroup iff the mapping $\lambda : a \mapsto a^\wedge$ is an injection from A into $A^\wedge$. (Then we can assume that $A^\wedge$ is a covering semigroup of $\underline{A}$ and it is now called the universal covering of $\underline{A}$.)

The cardinal number $|A_1|$ of the set A_1 of all elements $a^\wedge$ of $A^\wedge$, when $a \in A$, is called the semigroup order of the algebra $\underline{A}$, and if $|A_1| = 1$, then $\underline{A}$ is called a semigroup singular algebra.

1.3°. Every algebra is a subalgebra of a semigroup singular algebra ([4]).

It is easy to determine the universal enveloping semigroup of a semigroup singular algebra $\underline{A} = (A;F)$.

First, let J_F be the set of naturals defined by

$$J_F = \{n-1 \mid F_n \neq \emptyset\}, \qquad (1.1)$$

and let d_F be the greatest common divisor of the elements of J_F. Then:

1.4°. If $\underline{A}$ is a semigroup singular algebra, then $A^\wedge$ is the cyclic group with order d_F. ([4]).

It is natural to ask the question: what are the implications of the statement that the universal enveloping semigroup $A^\wedge$ of an algebra $\underline{A}$ has corresponding given properties. We do not think that (in general) a property of $A^\wedge$ gives a good information about the structure of A, but if $\underline{A}$ belongs to some special class of algebras (like polyadic semigroups or groups) we usually have a better situation.

§2. Associatives

The class of F-semigroups is a quasivariety and a convenient system of axioms (in the form of quasiidentities) of this quasivariety is given in [2]. Here we shall state some results concerning F-semigroups.

2.1°. The class of F-semigroups is a variety iff $d_F \in J_F$. (J_F and d_F are defined in §1, (1.1).) ([19])

An algebra $\underline{A} = (A;F)$ is called a weak associative if for any $f \in F_n$, $g \in F_m$ and $i \in \{2, \dots, m\}$ the following identities are satisfied:

$$gfx_1 \dots x_{m+n-1} = fgx_1 \dots x_{m+n-1} = gx_1 \dots x_{i-1} fx_i \dots x_{m+n-1}. \qquad (2.1)$$

And, a weak associative is called an associative if for any $f_\nu \in F_{n_\nu+1}$, $g_\lambda \in F_{m_\lambda+1}$, $\nu \in \{1, \dots, r\}$, $\lambda \in \{1, \dots, s\}$

such that

$$n_1 + n_2 + \dots + n_r = n = m_1 + m_2 + \dots + m_s,$$

the following identity is satisfied in $\underline{A}$:

$$f_1 \dots f_r x_o x_1 \dots x_n = g_1 \dots g_s x_o \dots x_n. \quad (2.2)$$

(In other words, $\underline{A}$ is a weak associative if continued products do not change by any replacement of operator symbols, and an associative if continued products with same length are equal.)

Some connections between the classes of F-semigroups, weak associatives and F-associatives are given in the next statements.

2.2°. Every weak F-associative is an F-associative iff $F = F_n = \{f\}$ consists of only one n-ary operator. (In that case, the associative $(A;f)$ is called an n-semigroup.) ([4])

2.3°. The class of F-semigroups is a subclass of the class of F-associatives. ([19])

2.4°. Every F-associative is an F-semigroup iff $d_F \in J_F$. (In this case, it may be assumed that an F-associative is, in fact, a d_F-semigroup.) ([19])

Example. Assume that $d_F \notin J_F$ and let n be the least element of J_F and p the least element of J_F which is not divisible by n. Define an algebra $\underline{A} = (\{0,1,2\};F)$ in the following way:

- if $g \in F_m$, $m \neq p+1$, then $g x_1 x_2 \dots x_m = 0$,
- if $f \in F_{p+1}$, then $f22 \dots 2 = 1$, and

$$f x_o x_1 \dots x_p = 0 \quad \text{when} \quad (x_o, \dots, x_p) \neq (\underbrace{2, \dots, 2}_{p+1}).$$

Then $\underline{A}$ is an associative but it is not an F-semigroup.

Thus, if $d_F \notin J_F$, then the class of F-semigroups is a proper subclass of the variety of F-associatives. Below we shall state some sufficient conditions under which an F-associative is an F-semigroup.

2.5°. If an F-associative $\underline{A}$ satisfies some of the following conditions, then it is an F-semigroup ([2] [20]):

(i) $\underline{A}$ is surjective, i.e.

$$(\forall a \in A)(\exists f \in F)(\exists a_1, \dots, a_n \in A) \quad a = f a_1 \dots a_n.$$

(ii) A is cancellative, i.e. for each n-ary operator $f \in F$, $i \in \{1,\dots,n\}$ and $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$ the mapping

$$x \mapsto f a_1 \dots a_{i-1} x a_{i+1} \dots a_n$$

is an injection.

An associative $\underline{A}$ is cyclic if it is generated by one of its elements. A description of the class of cyclic F-associatives is given in [3]. We note that the associative $\underline{A}$ in the above example is cyclic and it is generated by the element 2.

As concerns the singularity of associatives we have the following propositions.

2.6°. If an associative $\underline{A}$ is semigroup singular, then $|A| = 1$. ([4]).

2.7°. If $|F| \geq 2$, then there exists semigroup singular weak F-associatives; the class of subalgebras of semigroup singular weak F-associatives is a proper subclass of the class of weak F-associatives. ([4]).

§3. n-subsemigroups of semigroups

As we mentioned in 2.2°, an n-semigroup Q is an algebra (Q,f) with an associative n-ary operation f. Instead of $fx_1 \ldots x_n$, we shall write $[x_1 x_2 \ldots x_n]$. If $a_o, a_1, \ldots, a_{s(n-1)}$ (s≥1) is a sequence of elements of Q, then all continued products $\Pi(a_o, a_1, \ldots, a_{s(n-1)})$ are equal in Q, and thus we may denote by $[a_o a_1 \ldots a_{s(n-1)}]$ any such a product; if s=0, then we also write $[a_o] = a_o$. The universal covering semigroup Q^ of an n-semigroup Q can be characterized in the following manner ([7], [17], [18]):

3.1°. $Q\hat{} = Q_1 \cup Q_2 \cup \ldots \cup Q_{n-1}$, where $Q_1 = Q$, $Q_i = \{a_1 \ldots a_i \mid a_1, \ldots, a_i \in Q\}$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

3.2°. Two sequences $\underline{a} = (a_1, \ldots, a_i)$ and $\underline{b} = (b_1, \ldots, b_i)$ (a_ν, $b_\lambda \in Q$, $i \leq n$) are said to be strongly linked, then we write $\underline{a}\ s\ell\ \underline{b}$, iff there is a sequence $\underline{c} = (c_1, c_2, \ldots, c_t) \in Q^t$ and nonnegative integers $p_o, p_1, \ldots, p_i$, $q_o, q_1, \ldots, q_i$, such that

$$0=p_o<p_1<\ldots<p_i, \quad 0=q_o<q_1<\ldots<q_i, \quad p_i=q_i=t$$
$$p_{\nu+1}-p_\nu \equiv 1 \pmod{n-1}, \quad q_{\nu+1}-q_\nu \equiv 1 \pmod{n-1}, \qquad (3.1)$$
$$a_\nu=[c_{p_{\nu-1}+1} \ldots c_{p_\nu}], \quad b_\nu=[c_{q_{\nu-1}+1} \ldots c_{q_\nu}].$$

The transitive extension of $s\ell$ will be denoted by ℓ. Then, if $a_1, \ldots, a_i$, $b_1, \ldots, b_i \in Q$, the equation $a_1 \ldots a_i = b_1 \ldots b_i$ holds in Q iff $(a_1, \ldots, a_i)\ell(b_1, \ldots, b_i)$.

Below we shall state some connections between the properties of an n-semigroup Q and its universal covering semigroup Q^; it is assumed here that $n \geq 3$.

3.3°. The semigroup Q^ is commutative iff Q satisfies the following conditions:

(i) Q is commutative (i.e. for every permutation $\nu \to i_\nu$ of 1,2,...,n the identity $[x_1 \ldots x_n] = [x_{i_1} \ldots x_{i_n}]$ holds in Q),

(ii) $P = Q \setminus \{[a_1 \ldots a_n] \mid a_1, \ldots, a_n \in Q\}$ contains at most one element. ([18])

3.4°. Some covering semigroup of an n-semigroup Q is commutative iff Q is commutative. ([18])

3.5°. Some covering semigroup of an n-semigroup Q is cancellative iff Q is cancellative. ([15])

3.6°. Q^ is a group iff Q is an n-group (i.e.

$$(\forall a_1, \ldots, a_{n-1} \in Q)(\exists x, y \in Q)\{[xa_2 \ldots a_n] = a_1, [a_1 \ldots a_{n-1}y] = a_n\}). \quad ([17])$$

3.7°. Some covering semigroup of Q is a group iff Q is an n-group. ([1])

3.8°. Q is freely generated (in the variety of n-semigroups) by B iff Q^ is freely generated (in the variety of semigroups) by B. ([2;§4])

Assume now that $\mathcal{C}$ is a class of semigroups and denote by $\mathcal{C}(n)$ the class of n-semigroups which are n-subsemigroups of semigroups which belong to $\mathcal{C}$. We think that the question of description of $\mathcal{C}(n)$ is interesting when $\mathcal{C}$ is given in a convenient way.

As a consequence of a general result from the model theory ([14; p.274]) it follows that if $\mathcal{C}$ is a variety, then $\mathcal{C}(n)$ is a quasivariety. We do not known any convenient description of the set of varieties of semigroups such that $\mathcal{C}(n)$ is also a variety. We shall state here some partial results concerning this problem.

3.9°. Let $\mathcal{P}_{r,m}$ be the variety of semigroups that satisfy the identity $x^r = x^{r+m}$ and $\mathcal{C}_{r,m}$ the variety of commutative $\mathcal{P}_{r,m}$-semigroups. Then:

(i) $\mathcal{P}_{r,m}(n)$ is a variety iff n-1 is a divisor of m or $r \in \{0,1\}$.

(ii) $\mathcal{C}_{r,m}(n)$ is a variety for any r,m,n. ([16])

3.10°. Let $\mathcal{D}^{\ell}$ be the variety of left distributive semigroups (i.e. semigroups which satisfy the identity xyz = xyxz), and $\mathcal{D}$ the variety of distributive (both left and right) semigroups. Then $\mathcal{D}^{\ell}(n)$ is not a variety for any $n \geq 3$, and $\mathcal{D}(n)$ is a variety for any n. ([16])

Let $\xi = x_{i_1} x_{i_2} \ldots x_{i_p}$ be a (semigroup) term, where $i_\nu \in \{1,2,3,\ldots\}$. Then $|\xi|_i$ is the number of i_ν such that $i_\nu = i$.

3.11°. If $\mathcal{C}$ is a variety of semigroups defined by a set of identities $\xi = \eta$ such that $|\xi|_i \equiv |\eta|_i$ (mod n-1) for each $i = 1,2,\ldots$, then $\mathcal{C}(n)$ is a variety.

We note that in [5] it is given a description of the class $\mathcal{P}(n)$, where $\mathcal{P}$ is the class of periodic semigroups, and in [15] it is given a description of the class $\mathcal{G}_p(n)$ where $\mathcal{G}_p$ is the class of groups.

If $\mathcal{C}$ is a class of (binary) semigroups, it is natural to ask for a convenient set of classes, $\{\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n, \ldots\}$, such that $\mathcal{C} = \mathcal{C}_2$ and $\mathcal{C}_n$ is a class of n-semigroups. There are several possibilities of solution of that problem. Let us mention the following three of them:

(A) $Q \in \mathcal{C}_n'$ iff $Q^\wedge \in \mathcal{C}$,

(B) $Q \in \mathcal{C}_n''$ iff $S \in \mathcal{C}$ for some covering semigroup S of Q,

(C) $\mathcal{C}_n$ is defined directly.

It is clear that:

3.12°. $\mathcal{C}_n' \subseteq \mathcal{C}_n''$ for every class $\mathcal{C}$ of semigroups.

From 3.6°-3.8° it follows that (A), (B) and (C) give same solutions if $\mathcal{C}$ is the class of groups or the class of free semigroups. If $\mathcal{C}$ is the class of commutative semigroups (or the class of cancellative semigroups), then (B) and (C) give the same result, but the results of (B) and (A) are not identical (namely, $\mathcal{C}_n'$ is a proper subclass of $\mathcal{C}_n''$).

We note that if we try to define the notion of periodic n-semigroup by a definition of type (A) or (B) we would obtain unsatisfactory results. For example, if a ternary operation [xyz] is defined on $\mathbb{Z}$ by [xyz] = x-y+z, then we obtain a periodic ternary semigroup, but no covering semigroup of $(\mathbb{Z}; [\ ])$ is periodic.

We also note that the classes of completely regular and inverse n-semigroups are defined in [11] and [13] respectively by definitions of type (C).

§4. Universal covering groups

The well-known Post Coset Theorem ([8; p. 218]) gives a connection between the polyadic groups and binary groups.

The proposition 3.6° is, in fact, a modification of that theorem and, by this proposition, if Q is an n-group, then Q^ is a group and it is called the universal covering group of Q. We shall state here some known results (4.1°–4.7°) about Q^, assuming as in §3 that $n \geq 3$.

4.1°. Let Q^ be the universal covering of an n-group Q. Then

(i) If $a_1,\dots,a_i,\ b_1,\dots,b_i \in Q$, $1 \leq i \leq n-1$, then $a_1 \dots a_i = b_1 \dots b_i$ in Q^ iff there exist $c_1,\dots,c_{n-i} \in Q$ such that $[c_1\dots c_{n-1}a_1\dots a_i] = [c_1\dots c_{n-i}b_1\dots b_i]$.

(ii) If Q* is a covering semigroup of Q such that:

$$a_1,\dots,a_i,\ b_1,\dots,b_j \in Q \Rightarrow$$
$$\Rightarrow (a_1 \dots a_i = b_1 \dots b_j \text{ in } Q^* \Leftrightarrow i \equiv j \pmod{n-1}),$$

then Q* is the universal covering group of Q, Q^ = Q*.

(iii) Q_{n-1} (see 3.1°) is a normal subgroup of Q^ and the factor group $Q\text{\^{}}/Q_{n-1}$ is cyclic with order n-1.

(iv) If $a_1,\dots,a_k \in Q$, $\alpha_1,\dots,\alpha_k \in \mathbb{Z}$, then

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots a^{\alpha_k} \in Q_i \Leftrightarrow \alpha_1 + \dots + \alpha_k \equiv i \pmod{n-1}.$$

The proposition 3.6° is used in [1] for obtaining several axiom systems for n-groups. Some of them are:

4.2°. Let Q be an n-semigroup. The following statements are equivalent:

(i) Q is an n-group.

(ii) For some $\kappa \in \{1,2,\dots,n-2\}$,

$$(\forall x_1,\dots,x_k \in Q)\,(\exists x_1',\dots,x_{n-k-1}' \in Q)\,(\forall y \in Q)$$
$$[x_1\dots x_k x_1'\dots x_{n-k-1}' y] = y = [y x_1'\dots x_{n-k-1}' x_1\dots x_k].$$

(iii) There exists an (n-2)-ary operation $(\ )^{-1}$ on Q such that for any $x_1,\dots,x_{n-2}, y \in Q$ the following identity equalities hold:

$$[y x_1\dots x_{n-2}(x_1\dots x_{n-2})^{-1}] = y = [(x_1\dots x_{n-2})^{-1} x_1\dots x_{n-2} y].$$

(iv) There exists a unary operation $x \mapsto \bar{x}$ on Q such that the following identities are satisfied in Q:

$$[\bar{x}\, x^{n-2} y] = y = [y x^{n-2} \bar{x}].$$

(v) There exists a unary operation $x \mapsto \bar{x}$ on Q such that for some $p: 0 \leq p \leq n-2$ and for some $s: 0 \leq s \leq n-2$ the following identity equalities hold:

$$[x^p \bar{x}\, x^{n-p-2} y] = y = [y x^{n-s-2} \bar{x}\, x^s].$$

The notion of free n-group is defined in the class of n-groups in the usual manner, i.e. by a definition of the type (C) of §3. The situation in this case is not the same as in the class of n-semigroups which is evident from the following results 4.3°–4.6°, proved in [10].

4.3°. If Q is a free n-group with a basis B, then Q^ is a free group with the same basis B.

4.4°. If Q^ is a free group with rank $r \geq 2$, then Q is a free n-group.

4.5°. Q^ is an infinite cyclic group iff Q is isomorphic with the following n-subgroup of the additive group of integers:

$$\{(n-1)x + k \mid x \in \mathbb{Z}\} = A_k,$$

where $1 \leq k < n-1$ and k is relatively prime with n-1.

The n-group A_k is free iff $k = 1$ or $k = n-2$.

4.6°. An n-subgroup of a free n-group is a free n-group or an n-group isomorphic with an n-group of the form A_k.

We note that in the original proofs of the above results are used the corresponding "binary results", and the following property (see 1.1°) of n-groups ([7]):

4.7°. If $\phi : Q \to Q'$ is a monomorphism from an n-group Q into an n-group Q', then $\phi^\wedge : Q^\wedge \to Q'^\wedge$ is also a monomorphism. (In other words, if P is an n-subgroup of an n-group Q and if P* is the subgroup of $Q^\wedge$ generated by P, then P* is the universal covering group of P, i.e. $P^* = P^\wedge$.)

In the next propositions (4.8°–4.13°) are considered finite n-groups and it will be assumed there that Q is a finite n-group with order $q = |Q|$. First we have:

4.8°. The order of $Q^\wedge$ is $(n-1)\cdot q$.

Proof. By 3.1°, it suffices to prove that the set Q_i has q elements. Let $a_1,\dots,a_{i-1} \in Q$. Since (in $Q^\wedge$)

$$a_1\dots a_{i-1}x = a_1\dots a_{i-1}y \iff x = y \quad (x,y \in Q)$$

it follows that the set $a_1\dots a_{i-1}Q$ (which is a subset of Q_i) has q elements. On the other hand, if $b_1,\dots,b_i \in Q$ and $a_1\dots a_{i-1}x = b_1\dots b_i$, then $x = a_{i-1}^{-1}\dots a_{i-1}^{-1}b_1\dots b_i$ belongs to Q, from what follows that $Q_i = a_1\dots a_{i-1}Q$, i.e. $|Q_i| = q$. □

As an immediate corollary from 4.7° and 4.8° we obtain the following generalization of well known Lagrange Theorem for binary groups ([8; p. 222]):

4.9°. If P is an n-subgroup of Q, and $|P| = p$, then p is a divisor of q.

If $a \in Q$, then the order of the cyclic n-subgroup <a> generated by a is called the order of a in Q, and it is denoted by r(a). Denoting by $\hat{r}(a)$ the order of a in $Q^\wedge$, from 4.7°, 4.8° and 4.9° we obtain:

4.10°. $\hat{r}(a) = (n-1)r(a)$ and $r(a) \mid q$. □

Assume that $a \in Q$ and that $\bar{a}$ is the skew element of a (i.e. $\bar{a}$ is the solution x of the equation $[a^{n-1}x]=a$ in Q; see also 4.2°, (iv).) Then $\bar{a} = a^{2-n}$ (in $Q^\wedge$) and this implies that $\hat{r}(\bar{a})$ is a divisor of $\hat{r}(a)$. Therefore:

4.11°. $r(\bar{a})$ is a divisor of $r(a)$. □

In [9] it is considered the class of finite n-groups Q such that

$$(\forall a \in Q) \quad r(\bar{a}) = r(a).$$

Since

$$r(a) = r(\bar{a}) \iff \hat{r}(a) = \hat{r}(\bar{a}) = \hat{r}(a^{2-n}) \iff$$

$(\hat{r}(a), n-2) = 1$, it follows that:

4.12°. $r(a) = r(\bar{a})$ iff n-2 and r(a) are relatively prime.

We note that the main result of [9] proved without using the notion of covering group, is the following proposition:

4.13°. If n-2 and q are relatively prime, then $r(a) = r(\bar{a})$ for each $a \in Q$.

It is clear that 4.12° ⇒ 4.13°. Thus, a property of n-groups can be usually easier shown by an indirect method, i.e. by using the notion of the universal covering group, than by a direct method, i.e. by dealing only with n-groups. We shall give another example which support such a conjecture.

Let Q be an n-group (not necessarily finite) and let f and g be defined on $Q^\wedge$ by

$$f(x,y,z) = xy^{-1}z, \quad g(x_1,\dots,x_{n-2}) = (x_1x_2\dots x_{n-2})^{-1}.$$

Then

$$x_1x_2\dots x_n = f(x_1, g(x_2,\dots,x_{n-1}), x_n)$$

and $(Q;f,g)$ is a subalgebra of $(Q^\wedge;f,g)$.

Therefore:

4.14°. If Q is an n-group, then there is a ternary operation f and an n-2-ary operation g on Q such that

$$[x_1x_2\dots x_n] = f(x_1, g(x_2,\dots,x_{n-1}), x_n). \ \square$$

The proposition 4.14° is the main result of the paper [12], and it is proved there by the "direct method". We also note that by this "indirect method" can be proved some similar results on inverse n-semigroups which are obtained in [13].

§5 Universal covering semigroups of finite n-semigroups

It will be assumed in this section that Q is a finite n-semigroup and that $|Q| = q$, $n \geq 3$ are given. Clearly, $|Q_i| \leq q^i$ and by 3.1° we obtain:

5.1°. $Q^\wedge$ is finite and $|Q^\wedge| \leq q + q^2 + \dots + q^{n-1}$.

By 4.8° we have:

5.2°. If Q is an n-group, then $|Q_i| = q$ and $|Q^\wedge| = (n-1)q$.

This proposition sugests the question whether there exist finite n-semigroups with the property $|Q_i| = q$ which are not n-groups. The answer is positive. Namely,

let $Q = \{a, a^n, a^{2n-1}, \dots, a^{r(n-1)+1}, \dots, a^{(r+m-1)(n-1)+1}\}$ be a cyclic n-semigroup with index r and period m, where $q = r + m$. Then

$$a^{i(n-1)+1} = a^{j(n-1)+1} \iff i = j \text{ or } (i,j \geq r \text{ and } i \equiv j \pmod m)$$ and

$$Q^\wedge = \{a, a^2, \dots, a^{n-1}, a^n, \dots, a^{r(n-1)}, a^{r(n-1)+1}, \dots, a^{(r+m)(n-1)}\}$$

is a cyclic semigroup with index $r(n-1)$ and period $m(n-1)+1$. Therefore:

5.3°. If Q is cyclic, then $|Q_i| = q$, $|Q^\wedge| = (n-1)q$.

Now we shall prove that:

5.4°. If $P = Q \setminus \{[a_1\dots a_n] \mid a_1,\dots,a_n \in Q\}$ and $|P| = p$, then $|Q_i| \geq p^i + 1$.

Proof. First, if $a_1,\dots,a_i \in P$, then

$$(a_1,\dots,a_i)\, s\ell\, (b_1,\dots,b_i) \iff a_1 = b_1, \dots, a_i = b_i,$$

and this implies that $|Q_i| \geq p^i$. And, since the set $Q^i \setminus P^i$ is a union of ℓ-classes and is nonempty, we have also $|Q_i| \geq p^i + 1$. $\square$

As a corollary of 5.4° we obtain that:

5.5°. If Q is a constant n-semigroup, then

$$|Q_i| = (q-1)^i + 1.$$

Proof. Let $(\forall x_1,\dots,x_n)\ [x_1\dots x_n] = 0$. By 5.4° we have $|Q_i| \geq (q-1)^i + 1$, for $P = Q\setminus\{0\}$, $p = q-1$. Using the associativity of the operation [ ] and (5.1), it is easy to show that the following equality holds in $Q^\wedge$:

$$a_1 \dots a_{\nu-1} 0 a_{\nu+1} \dots a_i = 0^i$$

for any $a_1,\dots,a_{\nu-1}, a_{\nu+1},\dots,a_i \in Q$, $\nu \in \{1,\dots,i\}$, $i \in \{2,\dots,n\}$ and this implies that $|Q_i| = (q-1)^i + 1$. □

Let us consider the general case.

<u>5.6°</u>. If $q > 1$, then $1 < |Q_i| < q^i$ for each $i \in \{2,\dots,n-1\}$.

<u>Proof</u>. We have to prove that $1 \neq |Q_i|$, $|Q_i| \neq q^i$.

Assume that $|Q_i| = q^i$ for some $i \in \{2,\dots,n-1\}$. Thus, if $a_1,\dots,a_i, b_1,\dots,b_i \in Q$ and $a_1 \dots a_i = b_1 \dots b_i$, then $a_1 = b_1,\dots,a_i = b_i$. By the associativity we have:

$$[a_1 \dots a_n]\, a_1^{i-1} = a_1 \dots a_{i-1}[a_i \dots a_n a_1^{i-1}],$$

which implies that

$$[a_1 \dots a_n] = a_1$$

for any $a_1,\dots,a_n \in Q$. By symmetry we also have $[a_1 \dots a_n] = a_n$ and thus we have $a_1 = [a_1 \dots a_n] = a_n$ for any $a_1,\dots,a_n \in Q$. But this is impossible since $q = |Q| > 1$.

Assume now that $|Q_i| = 1$ for some $i \in \{2,\dots,n-1\}$. If $a,a_1,\dots,a_i, b,b_1,\dots,b_i \in Q$, then we have $a_1 \dots a_i = b_1 \dots b_i$ and also $ab_1 \dots b_{i-1} = bb_1 \dots b_{i-1}$, which implies

$$aa_1 \dots a_i = ab_1 \dots b_i = bb_1 \dots b_i,$$

i.e. $|Q_{i+1}| = 1$. Therefore we have $|Q_n| = 1$, i.e. Q is a constant n-semigroup and, by 5.5°, this implies that $|Q_i| = (q-1)^i + 1 > 1$. □

Denote by $\alpha(q,i,n)$ and $\alpha(q,n)$ the least positive integers such that there exists an n-semigroup Q such that

$$|Q| = q,\ |Q_i| = \alpha(q,i,n),\ |Q^\wedge| = \alpha(q,n).$$

Dually, $\beta(q,i,n)$ and $\beta(q,n)$ are the maximal numbers such that $|Q_i| = \beta(q,i,n)$, $|Q^\wedge| = \beta(q,n)$, for some n-semigroup Q with $|Q| = q$. Clearly:

<u>5.7°</u>. (i) $\alpha(1,i,n) = \beta(1,i,n) = 1$,

(ii) $\alpha(1,n) = \beta(1,n) = n-1$,

(iii) $\alpha(q,1,n) = \beta(q,1,n) = q$,

(iv) $\sum_{i=1}^{n-1} \alpha(q,i,n) \le \alpha(q,n) \le \beta(q,n) \le \sum_{i=1}^{n-1} \beta(q,i,n)$.

Assume that $q > 1$ and $1 < i \le n-1$. From 4.8° (5.3°) and 5.5° we obtain the following two propositions.

<u>5.8°</u>. $\alpha(q,i,n) \le q \le \beta(q,i,n)$,

$\alpha(q,n) \le (n-1)q \le \beta(q,n)$. □

<u>5.9°</u>. $\alpha(q,i,n) \le (q-1)^i + 1 \le \beta(q,i,n)$,

$\alpha(q,n) \le \sum_{i=1}^{n-1} (q-1)^i + n - 1 \le \beta(q,n)$.

<u>5.10°</u>. $2 \le \alpha(q,i,n) \le \beta(q,i,n) \le q^i - 1$,

$q + 2(n-2) \le \alpha(q,n) \le \beta(q,n) \le \sum_{i=1}^{n-1} q^i - (n-2)$.

Clearly (for $q > 2$), 5.8° gives a better approximation for $\alpha(q,n)$, and 5.9° a better approximation for $\beta(q,n)$.

We will make a remark about the decidability of $Q^\wedge$ if Q is a given finite n-semigroup. Namely, the description of $Q^\wedge$ given in §1 and §3 do not give a general procedure for obtaining the semigroup $Q^\wedge$. Namely, if $i \in \{2,\dots,n-1\}$, then there are infinitely many sequences of integers $p_0,\dots,p_i$, $q_0,\dots,q_i$ which satisfy (3.1) in 3.2°. But we can "improve" the description of $S\ell$ by the following proposition.

5.11°. If $a_1,\dots,a_i, b_1,\dots,b_i \in Q$ are such that $(a_1,\dots,a_i)\ s\ell\ (b_1,\dots,b_i)$, then there exist $(c_1,\dots,c_t) = \underline{c} \in Q^t$ and nonnegative integers $p_o, p_1,\dots, p_i, q_o, q_1,\dots,q_i$ such that

$$p_{\nu+1} - p_\nu \le n, \quad q_{\nu+1} - q_\nu \le n \qquad (5.1)$$

and (3.1) is satisfied.

Proof. Assume that (3.1) holds. If $p_1 \ge q_1,\dots,q_r$ and $p_1 < q_{r+1}$, then we can put:

$$a_1' = [b_1 \dots b_r\, c_{q_r+1} \dots c_{p_1}], \quad q_1' = \dots = q_r' = 1,$$

$$p_1' = 1 \text{ if } p_1 = 1 \text{ or } p_1' = n \text{ if } p_1 > n,$$

$$p_{\nu+1}' = p_\nu - q_r + r, \quad q_{r+\nu}' = q_{r+\nu} - q_r + \nu.$$

Thus, we can assume that

$$p_\nu - p_{\nu-1} \le n, \quad q_\lambda - q_{\lambda-1} \le n$$

for all $\nu, \lambda : 1 \le \nu \le s,\ 1 \le \lambda \le k$. In the same manner as above we can obtain $0 = p_o^*, p_1^*,\dots,p_i^*, 0 = q_o^*, q_1^*,\dots,q_i^*$ such that (3.1) is satisfied and

$$p_{\nu+1}^* - p_\nu^* \le n, \quad q_{\lambda+1}^* - q_\lambda^* \le n$$

for all $\nu, \lambda : 1 \le \nu \le s,\ 1 \le \lambda \le k$, and this will complete the proof.

From (5.1) it follows that $t \le in$, and thus we can decide if $(a_1,\dots,a_i)\ s\ell\ (b_1,\dots,b_i)$. And, as Q^i is finite, this implies a procedure for deciding if $(a_1,\dots,a_i)\ \ell\ (b_1,\dots,b_i)$. □

As it concerns finite n-groups, we have a "better" procedure. Namely, $Q_i = \{a^{i-1}x \mid x \in Q\}$, where a is a given element of Q, and if $a_1,\dots,a_i \in Q$, then the element $x \in Q$, such that $a^{i-1}x = a_1 \dots a_i$, is the solution of the equation $[a^{n-1}x] = [a^{n-i}a_1 \dots a_i]$.

§6. Presentation of n-semigroups and n-groups

Let B be a nonempty set and $F = F_B^{(n)}$ be the n-semigroup which is freely generated by B, i.e. F consists of all words $w = b_1 b_2 \dots b_p$ on B such that $p \equiv 1 \pmod{n-1}$, and the operation [ ] is defined on F by:

$$[w_1 w_2 \dots w_n] = w_1 w_2 \dots w_n. \qquad (6.1)$$

Let $\Lambda \subseteq F \times F$ and let Λ^* be the congruence on F generated by Λ. Then we say that the n-semigroup $Q = F/\Lambda^*$ "is given by the presentation $\langle B;\Lambda\rangle_n$" and, as usual, if $(u,v) \in \Lambda$, then it will be written $u = v$ instead of (u,v). It can be easily seen that:

6.1°. $\langle B;\Lambda\rangle_n^{\wedge} = \langle B;\Lambda\rangle_2$.

(Namely, $\langle B;\Lambda\rangle_2 = \langle B;\Lambda\rangle$ is a presentation in the class of semigroups.) □

We do not known the answer of the following question:

"If the presentation $\langle B;\Lambda\rangle_n$ is decidable, is the presentation $\langle B;\Lambda\rangle$ decidable too?"

This question is equivalent with the following one. Let Λ be as above and assume that the presentation $\langle B;\Lambda\rangle$ has the following property: there exists an effective procedure for deciding whether or not two words $u,v \in F$, such that $u = a_1 \dots a_p$, $v = b_1 \dots b_q$, $p \equiv q \pmod{n-1}$, define the same element in $\langle B;\Lambda\rangle$. "Is the presentation $\langle B;\Lambda\rangle$ decidable?"

Assume now that $B \ne \emptyset$, $B' = B \cup \{b^{-1} \mid b \in B\}$ and $F' = F_{B'}^{(n)}$ be the free n-semigroup generated by B'. Assume also that Σ is a set of words $w = b_1^{i_1} b_2^{i_2} \dots b_k^{i_k}$, $b_\nu \in B$, $i_\nu \in \mathbb{Z}$, such that

$$|w| = i_1 + i_2 + \dots + i_k \equiv 0 \pmod{n-1}.$$

Define a relation $\overset{\cdot}{\sim}$ on F' by:

$$u \overset{\cdot}{\sim} v \quad \text{iff} \quad (u=u_1bb^{-1}u_2,\ v=u_1u_2;\ \ b\in B) \quad \text{or}$$
$$(u = u_1b^{-1}bu_2,\ v=u_1u_2;\ \ b\in B) \quad \text{or}$$
$$(u=u_1wu_2,\ v=u_1u_2;\ \ w \in \Sigma).$$

If $\sim$ is the symmetric and transitive extension of $\overset{\cdot}{\sim}$, then it is a congruence on F' and the factor n-semigroup $F'/\sim$ is an n-group; we say that this n-group has a presentation $\langle B;\Sigma\rangle_{n,gp}$. Instead of $\langle B;\Sigma\rangle_{2,gp}$ we shall write $\langle B;\Sigma\rangle_{gp}$ and this is a presentation in the variety of groups. We have the following propositions:

6.2°. $\langle B;\Sigma\rangle_{n,gp} = \langle B;\Sigma\rangle_{gp}$. □

6.3°. The presentation $\langle B;\Sigma\rangle_{n,gp}$ is decidable iff the presentation $\langle B;\Sigma\rangle_{gp}$ is decidable.

Namely, if u and v are two "group words" on B, then we have first that

$$u = v \ \text{ in } \ \langle B;\Sigma\rangle_{gp} \Rightarrow |u| \equiv |v| \pmod{n-1},$$

and if $a\in B$ and k is a nonnegative integer such that $|a^ku| \equiv |a^kv| \equiv 1 \pmod{n-1}$, then

$$u = v \ \text{ in } \ \langle B;\Sigma\rangle_{gp} \ \text{ iff } \ a^ku=a^kv \text{ in } \langle B;\Sigma\rangle_{n,gp}. \ \square$$

R E F E R E N C E S

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