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On universal ramified sets

O univerzalnim granastim skupovima

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ON UNIVERSAL RAMIFIED SETS

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1. 1. There is no $\aleph_\alpha$-universal well-ordered set of cardinality $\aleph_\alpha$; the interval $I\omega_{\alpha+1}$ consisting of all ordinals $< \omega_{\alpha+1}$ is $\aleph_\alpha$-universal for well-ordered sets but is of cardinality $> \aleph_\alpha$. Any totally ordered set of cardinality $\leq \aleph_0$ which is $\aleph_0$-universal for well-ordered sets contains, isomorphically, the ordered set of rational numbers ([5], Theorem 1). Already G. Cantor ([1] p. 297) established that the set of rationals is $\aleph_0$-universal for ordered chains and is of cardinality $\aleph_0$.

The problem of universal ordered $\aleph_\alpha$-chains was studied by F. Hausdorff ([3], p. 182). E. g. the set $2(\omega_\alpha)$ of all the dyadic ω_α-sequences ordered alphabetically is an $\aleph_\alpha$-universal chain (of cardinality $2^{\aleph_\alpha}$ (cf. Hausdorff [3], p. 172—185, Sierpiński [7]). This means that any ordered chain of cardinality $\leq \aleph_\alpha$ is isomorph to a subset of $2(\omega_\alpha)$. Any Hausdorff's η_α-set H^α is $\aleph_\alpha$-universal for ordered chains $\leq \aleph_\alpha$.

The η_α-chains were particularly studied by L. Gillman [2].

1. 2. Any (partly or totally) ordered set O_α of cardinality $\leq \aleph_\alpha$ could be mapped chain-isomorphically in a one-to-one manner into any Hausdorff's η_α-set H^α; in other words, to any O_α corresponds some order destruction in H^α in order to get an order in H^α, denoted $H^\alpha(O_\alpha)$, in which O_α is isomorphically imbeddable (in particular: *incomparable* points of O_α are carried into incomparable points in $H^\alpha(O_\alpha)$). The question arises as to whether there exists a partial destruction of order in H^α such that *any* O_α be imbeddable into it. For $\alpha = 0$, the answer is in the affirmative.

1. 3. The question concerning $\aleph_\alpha$-universal ordered sets and quasi-ordered sets was studied by J. B. Johnston [4] and M. Novotný [6]. In particular, the ordinal power ${}^{\aleph_\nu \oplus \omega^*_\alpha}2$ is an $\aleph_\alpha$-universal quasi-ordered set (Novotný [6] 1., p. 328, Satz 1.)[1].

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[1] If X, Y are ordered sets, then the ordinal power ${}^XY$ consists of all the mappings $f: X \to Y$ ordered so that $f \leq g$ in ${}^XY$ means that $x \in X$ and $f(x) > g(x)$ imply $f(x') < g(x')$ for some $x' \in X$ satisfying $x' < x$. $\aleph_\alpha \oplus \omega_\alpha^*$ means the ordered sum of an antichain of cardinality $\aleph_\alpha$ and of type ω_α^*.

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1. 4. In this paper we are concerned with *universal ramifications* or ramified sets, i.e. with ordered sets R such that, for every $a \in R$, the ideal

$$R(.,a) = \{x \,;\, x \in R \wedge x < a\} \tag{1}$$

is a chain.

We shall see how the problem of $\aleph_\alpha$-universal ramified sets is connected with the Hausdorff's η_α-sets. In particular, the set of mappings of ideals of $I\omega_\alpha$ *into* η_α is a starting point for construction of universal $\aleph_\alpha$-ramified sets using a special extension of the $-|$ relation between functions.[2]

2. *An* $\aleph_0$*-universal ramified set. Set* H_{00}.

2. 1. Let us consider any $\aleph_0$-universal ordered chain H^0, e.g. the chain of rational numbers.

2. 2. Set $R(0)$. Let $R(0)$ be the set of all mappings f such that Dom f is an initial proper segment of $I\omega_0$ and Antidom $f \subseteq H^0$. Consequently, every member of $R(0)$ is a sequence of length $< \omega_0$ of points of H^0. Let $\gamma = \gamma s$, for any sequence s, denote the *length* or *height* of the sequence s; γs is an ordinal number.

We order the set $R(0)$ by the relation $\leq$ in this way: for $x, y \in R(0)$,

$$\begin{aligned} x \leq y \Longleftrightarrow x =| y \vee (\gamma x \leq \gamma y \wedge x(.,\beta) = \\ = y(.,\beta) \ (\beta < \gamma x) \wedge x_{\gamma x - 1} < y_{\gamma x - 1}), \end{aligned} \tag{1}$$

provided the number $\gamma x - 1$ exists. In other words, $x \leq y$ means that x is an initial segment of y or that $\gamma x \leq \gamma y$; in the last case for every proper initial segment x' of x one has $x' -| y$ as well as $x_{\gamma x - 1} < y_{\gamma x - 1}$ provided the ordinal $\gamma x - 1$ exists (for a sequence x and any ordinal $\beta < \gamma x$ the corresponding β-segment of x is denoted by $x(.,\beta)$).

2. 3. One proves readily that $(R(0); \leq)$ is a ramified set; in particular for any $a = (a_0, a_1, \ldots, a_n, \ldots)_{n < \gamma} \in R(0)$ the ideal $R(0)(.,a)$ has no last number and consists of all the points of the form $(a_0^<)$, $(a_0, a_1^<), \ldots, (a_0, a_1^< \ldots a_\nu^< \ldots), \ldots$, where for any $z \in H^0$ the symbol $z^<$ runs through the set $H^0(.,z)$ of points of H^0 that are $< z$.

2. 4. We stress particularly the chain $KR(0)$ of $R(0)$ consisting of all the single-point sets $\{x\}$, where $x \in H^0$.

2.5. I do not know whether already the set $R(0)$ is $\aleph_0$-universal; the trouble is that the set $R(0)$ contains the infima of some antichains A; in other words, the set $R(0)$ contains *neighbouring finite*

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[2] For functions f, g we write $f =| g$ or $g |= f$ provided Dom $f \subseteq$ Dom g and $g \mid \text{Dom} f = f$. By $f -| g$ we mean $f =| g$ and $f \neq g$.

parts inf A, A and consequently contains no point between inf A and all points of A. In order to remedy this we shall use a procedure of intercalation letting immediately follow every finite chain of the set by a replica of the ramification existing at that moment. E. g. if $a, b, c \in H^0$ and $a < b$ in H^0 then $(a) = \inf \{(b), (a, c)\}$ and $(b) \| (a, c)$.

2. 6. *Sets* $R^0, R^1 R^2, \ldots$. Let us constructs an ω_0-sequence of increasing ramified sets $R^0, R^1, \ldots$; we put $R^0 = R(0)$; for every chain C of R^0 satisfying $k C < \aleph_0$[3] let $R(C)$ be a replica of R^0 such that $R(C) = R(C')$ if C, C' are cofinal, and $R(C) \cap R(C') = \emptyset$ if C, C' are not cofinal. We let $R(C)$ immediatly follow C and consider the chain $K(R(C))$ of $R(C)$ (see **2. 4.**); we put it between C and the cone

$$R^0(C, .) = \{y ; y \in R^0 ; C < y\}, \tag{1}$$

in such a way that $R(C) \setminus K R(C)$ be incomparable with $R^0 \setminus R^0(., C]$, where

$$R^0(., C] = \{x ; x \in R^0 ; x \leq C\}. \tag{2}$$

We denote by R^1 the ramified set

$$R^0 \cup \bigcup_{C \subseteq R^0} R^0(C, .). \tag{3}$$

By repetition of the same operation we construct R^2, then R^3, etc.

2. 7. *The Universal set* H_{00}. The union of the sets R^ν $(\nu < \omega_0)$ is the requested set H_{00}:

$$H_{00} = \bigcup R^\nu, (\nu < \omega_0). \tag{1}$$

2. 7. 1. *The cardinality of* H_{00} *is* $\aleph_0$.

As a matter of fact each of the sets R^ν in (1) is countable; therefore the set H_{00} too has $\aleph_0$ points.

2. 7. 2. *The set* H_{00} *is ramified;* in particular, for every point x $(\in H_{00})$ the set $H_{00}(., x)$ is a chain containing no maximal member. If x, y are 2 distinct (incomparable) members of H_{00}, then $H_{00}(., x)$, $H_{00}(., y)$ are distinct sets (each containing points not belonging to the other one).

2. 7. 3. For every subset X of H_{00} satisfying $k X < \aleph_0$ the set

$$\cap_x H_{00}(., x) \quad (x \in X)$$

is a chain containing no last member; therefore, if X is not totally ordered the point inf X does not exist.

2. 7. 4. T h e o r e m. *The ramified set* H_{00} *contains isomorphically every ramified set* R *of cardinality* $\leq \aleph_0$, *i. e. every ramified set* $\leq \aleph_0$ *is imbeddable in* H_{00}.

[3] We could assume that $k C = 1$; the wording in the text is suited for $\alpha > 0$ too.

P r o o f. Let us well-order both R and H_{00}:

$$R = \{r_0, r_1, r_2, \ldots, r_m, \ldots\} \quad (m < \omega_{(kR)}) \tag{2}$$

$$H_{00} = \{h_0, h_1, h_2, \ldots, h_n, \ldots\}, \; n < \omega_0. \tag{3}$$

Here, $\omega_{(kR)}$ denotes the first ordinal number α such that $kI\alpha = kR, I\alpha = \{\xi; \xi < \alpha\}$.

We put $i_1(r_0) = h_0$. Let m be any ordinal of cardinality $< kR$ and suppose that the m-sequence of increasing isomorphisms

$$i_e \,|\, \{r_0, r_1, \ldots, r_j, \ldots\}_{j<e} \; (e < m)$$

into H_{00} is defined for every $e < m$; let us define the isomorphism $i_m \,|\, \{r_0, r_1, \ldots, r_{m-1}\}$ as the following extension of the isomorphism i_{m-1}. The domain of i_m is Dom $i_{m-1} \cup \{r_{m-1}\}$; we consider the following partition of Dom $i_{m-1} = D_{m-1}$ by the point r_m:

$$D_{m-1} = D_{m-1}(., r_m) \cup D_{m-1}(r_m, .) \cup CD_{m-1}[r_m]; \tag{4}$$

$$\text{we put } CD_{m-1}[r_m] = \{x;\; x \in D_{m-1},\; x \,\|\, r_m\}.$$

We define $i_m r_m$ as the *first* element h_n in the well-order of $H_{00} \setminus i_{m-1} D_{m-1}$ such that, corresponding to the relation (4), one has

$$i_m D_{m-1} = i_{m-1} D_{m-1}(., h_n) \cup i_{m-1} D_{m-1}(h_n, .) \cup CD_{m-1}[h_n]. \tag{5}$$

we put $CD_{m-1}[r_m] = \{x; x \in D_{m-1}, x \,\|\, r_m\}$.

The existence of such an h_n is implied by the following argument. The disjoint partition (4) of D_{m-1} implies the corresponding disjoint partition

$$i_m D_{m-1} = A \cup B \cup C, \text{ where} \tag{6}$$

$$A = i_{m-1} D_{m-1}(., r_m), \; B = i_{m-1} D_{m-1}(r_m, .), \tag{7}$$

$$C = i_{m-1} D_{m-1} \setminus (A \cup B).$$

In particular, h_n should satisfy the relations

$$A. < x < . B \text{ and } x \,|\, . C \tag{8}$$

(the sign $A\cdot$ or $\dot{A}$ or $\cdot A$ denotes every member of A).
Now, let us consider the sets

$A_0 = \bigcap_{a \in A} H_{00}(., a)$, $B_0 = \bigcap_{b \in B} H_{00}(., b)$, $C_0 = \bigcap_{c \in C} H_{00}(., c)$. The sets A_0, B_0, C_0 are chains in H_{00}, none having a last element (consequence of the fact that the cardinality of every of the sets A, B, C is $< \aleph_0$) (cf. **2.7.3**). A_0 is a proper initial part of B_0. Let us consider the set $E = B_0 \cap C_0$. E is a non empty chain, because the set $B \cup C$ is $< \aleph_0$. Since every member of B is incomparable to every member of C one concludes that the set E is a proper part both of B_0 and C_0, and the sets $B_0 \setminus C_0$, $C_0 \setminus B_0$ are non empty. In particular, the set $B_0 \setminus C_0$ is non empty; since the set A_0 is a proper initial portion of B_0, we conclude that the set

$$B_0 \setminus (A_0 \cup C_0) \tag{9}$$

is non empty; every point x of the set (9) satisfies (6) and yields the requested partition (6); therefore it is sufficient to take for x the first member h_n of the well-order (3) belonging to the set (9).

The existence of h_n is proved.

Finally one defines the mapping $i|R$ as the one which on $\{r_0, r_1, \ldots, r_j, \ldots\}_{j<m}$ equals i_m, for every $m < kR$. One sees that $i|R$ is an isomorphism of R into H_{00}. Q. E. D.

3. *Universal ramified sets $H_{\alpha\alpha}$.*

Starting from any η_α-set H^α one constructs in a way analogous to considerations in **2.** an $\aleph_\alpha$-universal ramified set $H_{\alpha\alpha}$; if $kH^\alpha \leq 2^{\aleph_\alpha}$ the cardinality $kH_{\alpha\alpha}$ of $H_{\alpha\alpha}$ equals kH^α for every non limit ordinal α; in particular, to Hausdorff's normal η_α-set (this set is of cardinality $2^{\aleph_{\alpha-1}}$) is associated the $\aleph_\alpha$-universal ramified set of cardinality $2^{\aleph_{\alpha-1}}$.

4. *On the $\eta_{\alpha\alpha}$-sets.* Let α be any ordinal number; an *$\eta_{\alpha\alpha}$-ordered set* is any ordered set O satisfying the following condition $C(\alpha)$:

Condition $C(\alpha)$. Any ordered subset of cardinality $< \aleph^\alpha$ admits in the set an extension in every direction. Precisely this means that for any ordered subset X of O such that $kX < \aleph_\alpha$ and for any ordered set X_0 such that $X_0 \supset X$ and $k(X_0 \setminus X) = 1$[4] there exists a point $p \in O$ depending on the single point x_0 of $X_0 \setminus X$ such that the identity mapping on X plus the mapping $x_0 \to p(x_0)$ be an isomorphism between the sets $X_0 = X \cup \{x_0\}$ and $X \cup \{p\}$.

If O denotes some special kind of ordered sets (like chains, ramified sets, lattices, etc.) the corresponding $C(\alpha)$-condition should be modified in an obvious way subjecting X_0 to belong to the same kind of orders.

5. Theorem. *Let α be any ordinal number; any two 1) totally ordered sets 2) ramified sets, 3) ordered sets, each of cardinality $\aleph_\alpha$ and satisfying the condition $C(\alpha)$ are isomorphic* (we do not know whether-except the case $\alpha = 0$-any such $\eta_{\alpha\alpha}$-set of cardinality $\aleph_\alpha$ exists).

Proof. Let S and T be any two $\eta_{\alpha\alpha}$-sets from the theorem. We might assume that S is disjoint from T. Let

$$S = (s_0, s_1, \ldots, s_\nu, \ldots)_{\nu<\omega_\alpha}, \tag{1}$$

$$T = (t_0, t_1, \ldots, t_\nu, \ldots)_{\nu<\omega_\alpha}, \tag{2}$$

[4] Irrespective whether the relation $X \subseteq O$ holds or does not hold.

be a normal well-order of S and of T respectively. We shall construct an isomorphism from S onto T as union of an increasing ω_α-sequence i_ν $(\nu < \omega_\alpha)$ of isomorphisms between some parts of S and T.

We define the isomorphism $i_0 \mid s_0 = t_0$, i. e. $s_0 = i_0^{-1} \mid t_0$. Let ν be any ordinal between 0 and ω_α and let us suppose that an increasing ν-sequence of isomorphisms $i_{\nu'}$ $(\nu' < \nu)$ with domains $< \aleph_\alpha$ is defined, so that any two consecutive domains differ by a single point. Let us define the isomorphism i_ν and its inverse i_ν^{-1} too.

If ν is a limit ordinal we define i_ν as the union of the mappings $i_{\nu'}$, i. e. $\mathrm{Dom}\, i_\nu = \bigcup \mathrm{Dom}\, i_{\nu'}$, and $i_\nu x = i_\xi x$, for every $x \leqslant \mathrm{Dom}\, i_\xi$. If ν has an immediate predecessor, let us consider the number $\nu - 1$.

I If $\nu - 1$ is even we define the mapping i_ν as the extension of $i_{\nu-1}$ satisfying the following two conditions:

Dom i_ν is the union of Dom $i_{\nu-1}$ and of the first member s in (1) belonging to the set $S \setminus \mathrm{Dom}\, i_{\nu-1}^{-1}$; we define $i_\nu s$ as the first member in the well-order (2) of T that does not belong to $\mathrm{Dom}\, i_{\nu-1}^{-1}$. The existence of the point $i_\nu s$ is implied by the fact that T satisfies the $C(\alpha)$-condition. Namely, we consider the ordered set $B_0 = B \cup \{s\}$, where $B = \mathrm{Dom}\, i_{\nu-1}^{-1}$. Since B is a subset of cardinality $< \aleph_\alpha$ in T and since T satisfies the $C(\alpha)$-condition, there exists a point p in $T \setminus B$ such that the mapping on $B \cup \{s\}$ equaling p in s and being identity on B is an isomorphism between $B \cup \{s\}$ and $B \cup \{p\}$; we determine p in a unique way by the additional condition that p be the first admissible point in the well-order (2) of T.

II If $\nu - 1$ is odd, we construct i_ν and i_ν^{-1} by reversing the role of the sets S, T in the case **I**.

The existence of i_ν being proved for every $\nu\, (< \omega_\alpha)$, the isomorphism $i : S \to T$ between S and T is proved.

5.1. Theorem 1. *Any ordered set O having the $C(\alpha)$ property is $\aleph_\alpha$-universal.*

2. *Every maximal subchain of O is a Hausdorff's η_α-set.*

The proof of **5.1.1.** runs like the proof of Theorem 5 (cf. the proof of Theorem 7).

As to **5.1.2.** cf. **6.1.**

6. *On an intercallation condition of ordered sets (or graphs).*

Let a be any cardinal number $\neq 0$; we say that an ordered set O has the *a-intercallation property* if for any ordered triplet (A, B, C) of subsets of O, each of cardinality $< a$, the conditions: every member of A precedes every member of B and every member of B is incomparable to every member of C, imply the existence of a point $p = p(A, B, C)$ in O such that

$$A \cdot < p < \cdot B \text{ and } p \parallel \cdot C; \quad \text{symbolically:}$$

$$(A, B, C \subseteq O) \wedge (kA, kB, kC < a) \wedge (A \cdot < \cdot B \parallel \cdot C) \Longrightarrow \Longrightarrow \bigvee_x x \leqslant O \wedge A \cdot < x < \cdot B \wedge x \parallel \cdot C. \tag{1}$$

The conclusion says that the subsets A, B, C respectively are in the left x-cone $R(., x)$ of R, in the right x-cone $R(x, .)$ and in the complement of the x-cone $R[x]$ of R respectively.[5]

6.1. Since by convention the empty set preceeds every set, the previous a-condition implies (put $A = \emptyset$) that R has no first point; analogously, if R has the a-property, R has no last point (put $B = \emptyset$); analogously, to every chain C in R satisfying $kC < a$ corresponds some $x \leqslant R$ and some $y \leqslant R$ such that $x < C < y$.

One proves readily that every maximal chain of R, provided R has the $\aleph_\alpha$-intercallation property, is a Hausdorff η_α-set.

7. Theorem 1. *Every ramified set R with the $\aleph_\alpha$-intercallation property contains isomorphically every ramified set R' of cardinality $\leq \aleph_\alpha$.*

2. *Any two ramified sets R, R' of cardinality $\aleph_\alpha$ and with the $\aleph_\alpha$-intercallation property are similar to each other (such sets exist at least for $a = 0$).*

3. *The $C(a)$ condition and the $\aleph_\alpha$-intercallation property are equivalent for any ramified set R.*

Proof of 7.1. Let $wR = (r_\nu)_\nu$ and $wR' = (r'_\sigma)_\sigma$ be a normal well-order of R and of R' respectively. One defines by induction the following mapping i of R' into R: let $ir'_0 = r_0$; let σ be any ordinal such that the isomorphism $i: wR'(., \sigma) \to R$ is defined; obviously

$$X \equiv wR'(., \sigma) \subseteq wR'. \tag{2}$$

If in (2) the sign $\subseteq$ means $=$, the procedure is finished: i is a requested isomorphism; if $\subseteq$ in (2) means $\subset$, we consider the point r_σ' and define ir_σ' as the first point $x = r_\nu$ such that

$A (= iX(., r_\sigma')) . < x < . X(i_\sigma', .) = B$ and $x \parallel . C$ where C denotes the set of all the points of X, each incomparable to r_σ'. Since the sets A, B, C are $< \aleph_\alpha$ each and since $A < B$, $B \parallel C$, the $\aleph_\alpha$-property of R implies the existence of the requested point r_ν.

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[5] The same statement could be formulated for oriented graphs; for a symmetrical graph (G, ϱ) the a-intercallation property would read as follows: $A \subseteq G \wedge B \subseteq G \wedge C \subseteq G \wedge A \cdot \varrho \cdot B \wedge B \cdot \varrho' \cdot C \wedge |A, B, C| < a \Longrightarrow \Longrightarrow \bigvee_x x \in G \wedge A \cdot \varrho x \varrho \cdot B \wedge x \varrho' \cdot C.$

The proof of Theorem 7.2 runs like the proof of Theorem 5.

Proof of Theorem 7.3. Since obviously every ramified set with $C(\alpha)$-property has also the $\aleph_\alpha$-intercallation property, let us prove the converse statement: If a ramified set R has the $\aleph_\alpha$-intercallation property, then R has the $C(\alpha)$ property, i. e. R is a $\eta_{\alpha\alpha}$-set.

Now let X be any subset of R of cardinality $<\aleph_\alpha$; let q be a point such that $q \notin X$ and that the set $X_0 = X \cup B \cup \{q\}$ is ramified; then we have the decomposition $X = A \cup B \cup C$, where $A = X(., q)$, $B = X(q, .)$, $C = X \setminus A \setminus B$.

The set R having the $\aleph_\alpha$-intercallation property, there exists some point p satisfying the relations (1) in **6**; the set $X \cup \{p\}$ as a subset of R is an extension of X such that the identity mapping on X plus the mapping $p \to q$ be an isomorphism between $X \cup \{p\}$ and $X \cup \{q\}$.

8. Remark. It should be interesting to produce an ordered set with the $\aleph_\alpha$-intercallation property but not having the $C(\alpha)$-property.

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O UNIVERZALNIM GRANASTIM SKUPOVIMA

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1.1. Nema $\aleph_\alpha$ — univerzalna dobro uređena skupa s $\aleph_\alpha$ članova (tj. nema dobro uređena skupa od $\aleph_\alpha$ članova u koji bi se svaki dobro uređen skup sa $\leq \aleph_\alpha$ članova mogao izomorfno smjestiti). Problem o $\aleph_\alpha$ — univerzalnim uređenim lancima izučavao je Hausdorff [3] p. 182, pa je npr. skup $2(\omega_\alpha)$ svih dijadskih ω_α — nizova uređen alfabetski određen $\aleph_\alpha$ — univerzalni lanac. Svaki Hausdorffov η_α — skup H^α je $\aleph_\alpha$ — univerzalan za uređene lance.

1.2. Svaki uređen skup O_α kardinalnosti $\leq \aleph_\alpha$ može se preslikati obostrano jednoznačno u svaki η_α — skup H^α tako da svaki lanac L iz O_α prelazi u slični lanac u H^α. Može li se u H^α poredak djelomično uništiti tako da se O_α može uroniti u tako preuređen skup? Ako je $\alpha = 0$, odgovor je potvrdan.

1. 3. Uređene i kvazi-uređene skupove koji su $\aleph_\alpha$ — univerzalni izučavali su Johnston [4] i Novotný [6].

1. 4. U ovom članku promatrat ćemo univerzalne granaste skupove tj. skupove za koje je svaki skup oblika (1) posve uređen.

2. *O jednom $\aleph_0$ — univerzalnom granastom skupu. Skup H_{00}.*

2. 2. — **2. 5.** Neka je $R(0)$ skup svih preslikavanja f kojima je oblast pravi segment od $I\omega_0$ a protuoblast je u H^0, gdje je H^0 bilo koji $\aleph_0$ — univerzalan lanac. Skup $R(0)$ uređujemo relacijom $\leq$ definiranom u (1) pa se vidi da je $R(0)$ granast i u njemu se ističe lanac $KR(0)$ sastavljen od svih jednočlanih članova iz H_α. Ne znamo da li je možda već skup $(R(0), <) \aleph_0$ — univerzalan granast skup.

2. 6. Za svaki lanac C iz $R(0)$ potencije $< \aleph_0$ promatra se primjerak $R(C)$ od $R^0 = R(0)$ tako da bude $R(C) = R(C')$, odnosno $R(C) \cap R(C') = \emptyset$, već prema tome da li su C, C' kofinalni ili nisu kofinalni; zatim stavimo $R(C)$ između C i konusa (1), tako da $R(C) \setminus \setminus KR(C)$ bude neusporedljiv sa $R^0 \setminus (2)$. Neka R^1 označuje granast skup (3). Slično se dalje definiraju $R^2, R^3, \ldots$.

2.7. Skup H_{00} je prema (1) unija skupova R^ν.

2.7.4. *Teorem. Skup H_{00} je granast skup od $\aleph_0$ tačaka i obuhvaća izomorfno svaki granast skup potencije $\leq \aleph_0$.*

Dokaz se provodi izgradnjom izomorfizma kao unije uzlazna niza izomorfizama.

3. Univerzalni granasti skupovi $H_{\alpha\alpha}$ izgrađuju se polazeći od η_α-skupa slično kao što je bilo za $\alpha = 0$.

4. $\eta_{\alpha\alpha}$-skupovi. To su uređeni skupovi za koje vrijedi uslov $C(\alpha)$: svaki uređen skup potencije $< \aleph_\alpha$ dopušta proširenje u svakom smjeru.

5. *Teorem. Neka je α redan broj; bilo koja dva skupa koja su: 1) potpuno uređena, 2) granasta, 3) uređena, a potencije su $\aleph_\alpha$ i zadovoljavaju uslov $C(\alpha)$ međusobno su izomorfna* (osim slučaja $\alpha = 0$, nepoznato je da li postoji $\eta_{\alpha\alpha}$-skup od $\aleph_\alpha$ tačaka).

5.1. *Teorem. (i) Svaki uređen skup O sa $C(\alpha)$ svojstvom je $\aleph_\alpha$-univerzalan. (ii) Svaki maksimalan lanac iz O je Hausdorffov η_α-skup.*

6. *O jednom svojstvu uklapanja uređenih skupova i orijentiranih grafova.*

Neka je a glavni broj $\neq 0$; kazat ćemo da uređen skup O ima svojstvo a-uklapanja, ako za svaku uređenu trojku (A, B, C) podskupova od O sa svojstvima: 1) A, B, C su potencije $< a$, 2) svaki član od A prethodi svakom članu od B i 3) svaki član od B je neusporedljiv sa svakim članom iz C, postoji tačka $p = p(A, B, C)$ iz

O koja je između A i B a neusporedljiva je sa svakim elementom iz C[1].

7. *Teorem* 1. *Svaki granast skup R sa svojstvom* $\aleph_\alpha$ *-uklapanja sadržava izomorfno svaki granast skup kardinalnosti* $\leq \aleph_\alpha$;

2. *Bilo koja dva granasta skupa R, R' kardinalnosti* $\aleph_\alpha$ *i sa svojstvom* $\aleph_\alpha$ *-uklapanja međusobno su izomorfna.*

3. $C(\alpha)$ *-svojstvo i svojstvo* $\aleph_\alpha$ *-uklapanja međusobno su dva ekvivalentna svojstva za granaste skupove.*

8. *Primjedba.* Bilo bi zanimljivo definirati uređen skup sa svojstvom $\aleph_\alpha$ -uklapanja a bez $C(\alpha)$ -svojstva.

(Primljeno 24. XII 1962.)

[1] Slično se definira svojstvo α-uklapanja za orijentirane grafove.